EVALUATION OF THREE FIRST-ORDER ISOTROPIC VISCOELASTIC FORMULATIONS BASED ON THE GENERALIZED STANDARD LINEAR SOLID

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ABSTRACT

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Stress and strain relaxation times in the generalised standard linear solid (GSLS) define the Q behavior as a function of frequency. Given Q_P and Q_S models, strain relaxation times are different for P- and S-waves; P- and S-wave stress relaxation times within each mechanism can be the same, when the τ method is used to estimate relaxation times from Q with multiple relaxation mechanisms, but can also be different (for example, when a single relaxation mechanism is used for both P- and S-waves). Both Robertsson's and Carcione's viscoelastic formulations are based on the GSLS. In Robertsson's formulation, the same stress relaxation times are used for both P- and S-waves; in Carcione's formulation, P- and S-wave stress relaxation times can be different. Moreover, the physical meanings of the P-wave stress and strain relaxation times in these two formulations are not the same. They are calculated from the P-wave quality factor, and the bulk quality factor, respectively; i.e., Robertsson's and Carcione's formulations have different parameterisations. To demonstrate that they are equivalent when the P- and S-stress relaxation times are the same, we derive the second-order viscoelastic equations in the frequency domain. We then generalise Robertsson's formulation to allow different stress relaxation times for P- and S-waves, by reformulating the memory variable equations. The generalised Robertsson's formulation is equivalent to Carcione's formulation. The seismograms modelled by Carcione's, Robertsson's and the generalised Robertsson's formulations are identical, when the input stress relaxation times are the same for P- and S-waves. With different P- and S-wave stress relaxation times in the input model, seismograms produced by Carcione's and the generalised Robertsson's formulations are indistinguishable, while there are obvious differences with seismograms produced by Robertsson's formulation. The limitation of using the same stress relaxation times for P- and S-waves in Robertsson's formulation produces the differences, and leads to errors in the effective Q modelled in the seismograms. Considering the accurate inclusion of intrinsic attenuation, Carcione's and the generalised Robertsson's formulation are equivalent choices as they both allow different P- and S-wave stress relaxation times. Robertsson's original formulation is equivalent only when the P- and S-wave stress relaxation times within each mechanism are the same. The three formulations have comparable computational cost.

KEY WORDS: stress and strain relaxation times, standard linear solid, viscoelasticity.

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INTRODUCTION

Seismic intrinsic attenuation can be significant, especially for reservoirs with fluid saturated fractures and pores, or with gas clouds (Maultzsch et al., 2007; Mangriotis et al., 2013). Velocity dispersion and amplitude loss (Q effects) strongly affect the waveforms (Kang and McMechan, 1994; Yang et al., 2015). Without properly including Q effects, errors may be introduced during seismic modeling, imaging and inversion (Liao and McMechan, 1996; Tiwari and McMechan, 2007; Prieux et al., 2013; Guo et al., 2016).

Q effects can be included using different rheological or mathematical models during wavefield extrapolations (Liu et al., 1976; Kjartansson, 1979; Ferry, 1980; Mozco and Kristek, 2005; Carcione, 2007; Moczo et al., 2007), including the generalized standard linear solid (GSLS) and the generalized Maxwell body (GMB). To incorporate intrinsic attenuation in time domain seismic modeling, the time convolution in the stress-strain relation of the attenuative media is removed by the introduction of memory variables (Day and Minster, 1984; Emmerich and Korn, 1987; Carcione et al., 1988a; Day, 1998; Kristek and Moczo, 2003). Day and Minster (1984) approximated the viscoelastic modulus with a low-order rational function of frequency, the coefficients of which can be determined by a Padé approximant method. Emmerich and Korn (1987) proposed to use the GMB to approximate the viscoelastic modulus, with improved accuracy compared to the Padé approximation. Following Liu et al. (1976), Carcione et al. (1988a,b), used the GSLS to approximate constant Q. Moczo and Kristek (2005) proved that GSLS and GMB are equivalent. In this study, we focus on the viscoelastic formulations based on the GSLS. Different formulations have been developed with superposition of relaxation mechanisms using the GSLS, including Carcione (1993); Robertsson et al. (1994); Xu and McMechan (1995); Hestholm, (2002) and recently, Yang et al. (2015).

In the following sections, the first-order viscoelastodynamic equations published by Carcione (1993) and Robertsson et al. (1994), are referred to as Carcione's formulation and Robertsson's formulation, respectively. Xu and McMechan (1995) introduced composite memory variables in a second-order displacement-memory variable formulation to improve the computational efficiency of Carcione's formulation. Hestholm (2002) gave the velocity-stress formulation for a curved grid based on Robertsson's formulation. The first-order formulation of Yang et al. (2015) is equivalent to Robertsson's formulation. The stress and strain relaxation times in the GSLS define the effective Q modelled in the viscoelastic formulations. When multiple relaxation mechanisms are used to fit constant Q behaviour as a function of frequency, they can be calculated by the τ method (Blanch et al., 1995; Hestholm et al., 2006), or by nonlinear optimization with a positivity constraint (Blanc et al., 2016). When only one relaxation mechanism is used, the stress and strain relaxation times can be calculated analytically from formulations of a standard linear solid (Carcione et al., 2007).

Yang et al. (2015) compared the viscoelastic formulation that they proposed, with Carcione's formulation, using common P and S-wave stress relaxation times. They showed there are obvious differences between seismograms calculated by the two viscoelastic formulations. They reformulated Carcione's algorithm to a form similar to theirs, compared the two formulations, and ascribed the seismogram differences to the asymmetry of the normal Z-component stress with the normal X- and Y-component stress in Carcione's formulation. However, it is not correct to directly compare Carcione's formulation and the first-order equations in Yang et al. (2015), since the physical meanings of the P-wave stress and strain relaxation times are not the same. In Carcione's formulation, the P-wave stress and strain relaxation times are calculated from the bulk quality factor Q_κ (Carcione et al., 1988c), and in Robertsson's formulations and the equations in Yang et al., (2015), they are calculated from the P-wave quality factor Q_P (Robertsson et al., 1994; Yang et al., 2015). The relation between Q_P and Q_κ is given by Savage et al. (2010).

One of the main limitations in the formulations of Robertsson et al. (1994) and Yang et al. (2015) is that the same stress relaxation time is used, within each mechanism, for both P- and S-waves. This assumption is valid, when the τ method (Blanch et al., 1995; Hestholm et al., 2006)) is used to calculate the relaxation times, which is common for viscoelastic modeling with multiple relaxation mechanisms, in which a $Q(\omega)$ behaviour is fitted across logarithmically-spaced frequencies. The stress relaxation time for the l-th mechanism $\tau_{\sigma l} = 1/\omega_l$, where each ω_l is a selected representative angular frequency (Blanch et al., 1995). However, when using the recently proposed positivity preserving method (Blanc et al., 2016) for estimating relaxation times with multiple relaxation mechanisms, or when only one relaxation mechanism is used, both the stress and strain relaxation times depend on Q, and thus the P- and S-stress relaxation times are usually not the same. With different P- and S-stress relaxation times as the input model, Robertsson's formulation is no longer applicable. Using one relaxation mechanism is common when approximating constant Q, in a narrow frequency band, to improve computational efficiency (Carcione, 2007).

In this study, we demonstrate, when using the same stress relaxation times for P- and S-waves, that Carcione's formulation is equivalent to Robertsson's formulation, by deriving the second-order viscoelastic equations in the frequency domain. We generalise Robertsson's formulation to allow different P- and S-wave stress relaxation times, by which we wish to provide correct viscoelastic modeling results when the P- and S-wave stress relaxation times are different in the model. Separate memory variables associated with P- and S-wave modulus, respectively, are used in each stress component, instead of composite memory variables. Numerical examples are given for viscoelastic modeling with the same (multiple relaxation mechanisms for P- and S-waves), and different, P- and S-wave stress relaxation times (one relaxation mechanism for P-waves, and one for S-waves) in the model, for approximating Q_P and Q_S. We compare the seismograms calculated by Carcione's, Robertsson's, and the generalized Robertsson's formulations. Q_P and Q_S are measured as a function of frequency (logarithmically-spaced) using the spectral ratio method (Båth, 1974; Kang and McMechan, 1994) from the direct waves in the seismograms. Computational cost is also considered. The aim of this study is to resolve the apparent inconsistencies between the three viscoelastic formulations, and to understand their consequences, and thus be able to use each of them appropriately.

ROBERTSSON'S AND CARCIONE'S VISCOELASTIC FORMULATIONS

The viscoelastic equations of Robertsson and of Carcione are shown in Appendices A and B, respectively. The formulation in Carcione (1993) cannot be directly reduced to 2D, since a linear combination of normal X- and Y-component memory variables [e_{11l} and e_{22l} in Carcione et al. (1988a)] is used to introduce the Q effects in the normal (Z)-component stress equation. We introduce the Z-component memory variable (e_{33l}), and replace the linear combination of e_{11l} and e_{22l} with e_{33l} in the Z-component stress-strain equation (Xu and McMechan, 1995).

In this section, we derive the second-order vector displacement equations in the frequency domain for both Robertsson's and Carcione's formulations, and use them to discuss the equivalence of the two formulations. Throughout this paper, in (the generalized) Robertsson's formulation, we use 'P' and 'S' labels to denote P- and S-waves, respectively; and we use '1' and '2' labels to denote P- and S-waves in Carcione's formulation.

Robertsson's formulation

For Robertsson's formulation, to derive the second-order viscoelastic equations, with displacement instead of particle velocity, we reformulate eqs. (A-1) to (A-5) as

$$\sigma_{ij} = \pi^U \nabla \cdot \mathbf{u} - 2\mu^U(\nabla \cdot \mathbf{u} - \partial_j u_i) + \sum_{l=1}^{L} r_{ij}^l, \quad \text{for } i = j, \tag{1}$$

$$\sigma_{ij} = \mu^U(\partial_j u_i + \partial_i u_j) + \sum_{l=1}^{L} r_{ij}^l, \quad \text{for } i \neq j, \tag{2}$$

$$\dot{r}^l_{ij} = -(1/\tau_{\sigma l})r^l_{ij} - (1/\tau_{\sigma l})\{(\pi/L)[(\tau^P_{\varepsilon l}/\tau_{\sigma l})-1]\nabla\cdot\mathbf{u}$$

$$- 2(\mu/L)[(\tau^S_{\varepsilon l}/\tau_{\sigma l})-1](\nabla\cdot\mathbf{u}-\partial_j u_i)\}, \quad \text{for } i = j, \text{ and } l = 1,\ldots,L, \tag{3}$$

$$\dot{r}^l_{ij} = -(1/\tau_{\sigma l})r^l_{ij} - (1/\tau_{\sigma l})(\mu/L)[(\tau^S_{\varepsilon l}/\tau_{\sigma l})-1](\partial_i u_j + \partial_j u_i),$$

$$\text{for } i \neq j, \text{ and } l = 1,\ldots,L, \tag{4}$$

and

$$\rho\ddot{u}_i = \partial_j\sigma_{ij}, \tag{5}$$

where σ_{ij} are stress components, r^l_{ij} is the memory variable component for the l-th relaxation mechanism, i,j = x,y,z, and $\mathbf{u}$ is the displacement vector with components u_i, for i = x,y,z; $\pi^U = \pi\{1 + (1/L)\Sigma^L_{l=1}[(\tau^P_{\varepsilon l}/\tau_{\sigma l})-1]\}$, $\mu^U = \mu\{[1 + (1/L)\Sigma^L_{l=1}[(\tau^S_{\varepsilon l}/\tau_{\sigma l})-1]\}$, and the letter 'U' refers to 'unrelaxed' (viscoelastic) parameters. $\pi = \lambda + 2\mu$, λ and μ are relaxed (elastic) Lamé constants, and ρ is mass density. L is the number of relaxation mechanisms for both P- and S-waves, $\tau_{\sigma l}$ is the stress relaxation time for the l-th mechanism for both P- and S-waves, and $\tau^P_{\varepsilon l}$ and $\tau^S_{\varepsilon l}$ are strain relaxation times for the l-th mechanism for P- and S-waves, respectively. The values of the relaxation times are space dependent. The dot and double-dot over variables indicate first-order and second-order time derivatives, respectively.

Applying a Fourier transform over time to the memory variable eqs. (3) and (4), we get

$$(i\omega\tau_{\sigma l} + 1)\tilde{r}^l_{ij} = \{-(\pi/L)[(\tau^P_{\varepsilon l}/\tau_{\sigma l})-1]\nabla\cdot\tilde{\mathbf{u}}$$

$$- 2(\mu/L)[(\tau^S_{\varepsilon l}/\tau_{\sigma l})-1](\nabla\cdot\tilde{\mathbf{u}}-\partial_j\tilde{u}_i)\} \quad , \text{ for } i = j, \text{ and } l = 1,\ldots,L, \tag{6}$$

and

$$(i\omega\tau_{\sigma l} + 1)\tilde{r}^l_{ij} = -(\mu/L)[(\tau^S_{\varepsilon l}/\tau_{\sigma l})-1](\partial_i\tilde{u}_j+\partial_j\tilde{u}_i),$$

$$\text{for } i \neq j, \text{ and } l = 1,\ldots,L, \tag{7}$$

where $\tilde{\mathbf{u}}$, $\tilde{u}_i$ and $\tilde{r}^l_{ij}$ are the Fourier transforms of $\mathbf{u}$, u_i and r^l_{ij}, ω is the angular frequency, and $i = \sqrt{-1}$.

Eqs. (6) and (7) have structures similar to eqs. (1) and (2). Fourier transforming eqs. (1), (2) and (5), and with algebraic operations to eliminate $\tilde{r}^l_{ij}$ and $\tilde{\sigma}_{ij}$, we obtain the second-order derivative displacement version corresponding to Robertsson's first-order particle velocity and stress formulation [eqs. (A-1) to (A-5)]

$$-\rho\omega^2\tilde{\mathbf{u}} = \nabla\Big\{\pi\Big[1 + (1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(\tau_{\varepsilon l}^{\mathrm{P}}/\tau_{\sigma l})-1]$$

$$- (1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}\{[(1/(\mathrm{i}\omega\tau_{\sigma l}+1)][(\tau_{\varepsilon l}^{\mathrm{P}}/\tau_{\sigma l})-1]\}\Big]\nabla\cdot\tilde{\mathbf{u}}\Big\}$$

$$- \nabla\times\Big\{\mu\Big[1 + (1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(\tau_{\varepsilon l}^{\mathrm{S}}/\tau_{\sigma l})-1]$$

$$- (1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}\{[(1/(\mathrm{i}\omega\tau_{\sigma l}+1)][(\tau_{\varepsilon l}^{\mathrm{S}}/\tau_{\sigma l})-1]\}\Big]\nabla\times\tilde{\mathbf{u}}\Big\}$$

$$= \nabla\{\pi(1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(1+\mathrm{i}\omega\tau_{\varepsilon l}^{\mathrm{P}})/(1+\mathrm{i}\omega\tau_{\sigma l})]\nabla\cdot\tilde{\mathbf{u}}\}$$

$$- \nabla\times\{\mu(1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(1+\mathrm{i}\omega\tau_{\varepsilon l}^{\mathrm{S}})/(1+\mathrm{i}\omega\tau_{\sigma l})]\nabla\times\tilde{\mathbf{u}}\} \ . \qquad (8)$$

From eq. (8), the P- and S-wave components can be projected as

$$-\rho\omega^2\tilde{\mathbf{u}}_{\mathrm{P}} = \nabla\{\pi(1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(1+\mathrm{i}\omega\tau_{\varepsilon l}^{\mathrm{P}})/(1+\mathrm{i}\omega\tau_{\sigma l})]\nabla\cdot\tilde{\mathbf{u}}\} \ , \qquad (9)$$

and

$$-\rho\omega^2\tilde{\mathbf{u}}_{\mathrm{S}} = -\nabla\times\{\mu(1/\mathrm{L})\sum_{l=1}^{\mathrm{L}}[(1+\mathrm{i}\omega\tau_{\varepsilon l}^{\mathrm{S}})/(1+\mathrm{i}\omega\tau_{\sigma l})]\nabla\times\tilde{\mathbf{u}}\} \ . \qquad (10)$$

If we select $\tau_{\sigma l} = \tau_{\varepsilon l}^{\mathrm{P}} = \tau_{\varepsilon l}^{\mathrm{S}}$ in eq. (8), and inverse Fourier transform eq. (8) to the time domain, we have

$$\rho\ddot{\mathbf{u}} = \nabla(\pi\nabla\cdot\mathbf{u}) - \nabla \times (\mu\nabla \times \mathbf{u}) \ , \qquad (11)$$

which is the second-order wave equation in an elastic medium (Aki and Richards, 1980).

Carcione's formulation

The difference between Carcione's and Robertsson's formulation is in their parameterizations. Carcione (1993) uses the bulk quality factor Q_κ, instead of Q_P, to introduce P-wave attenuation. Thus the P-wave stress and strain relaxation times in Carcione's formulation have different physical meanings than those in Robertsson's formulation. To evaluate the equivalence of the two formulations, we now derive the second-order displacement equation corresponding to Carcione's first-order formulation [eqs. (B-1) to (B-6) in Appendix B].

We use the equations of motion [eq. (A-10)] and the stress-strain relation [eq. (A-12)] in Appendix A of Carcione et al. (1988c), which are

$$\tilde{\sigma}_{ij,j} + \omega^2 \rho \tilde{u}_i = 0 \ . \tag{12}$$

and

$$\tilde{\sigma}_{ij} = (1/D)(M_1^C - M_2^C)\delta_{ij}\tilde{\varepsilon}_{kk} + M_2^C\tilde{\varepsilon}_{ij} \ , \tag{13}$$

where $\sigma_{ij,j}$ is the derivative of σ_{ij} in the j space direction, ε_{ij} are the normal (i=j) and shear (i $\neq$ j) strain components, and $\tilde{\sigma}_{ij}$, $\tilde{u}_i$ and $\tilde{\varepsilon}_{ij}$ are the Fourier transforms of σ_{ij}, u_i, and ε_{ij}, respectively. $\delta_{ij} = 1$, when i = j, and $\delta_{ij} = 0$, when i $\neq$ j. M_1^C and M_2^C are complex moduli, associated with P- and S-waves, respectively.

With the definition of the complex Lamé constants $\lambda^C = (1/D)(M_1^C - M_2^C)$ and $\mu^C = ½M_2^C$ (Carcione et al., 1988c), eq. (13) can be transformed into

$$\tilde{\sigma}_{ij} = \lambda^C\delta_{ij}\partial_j u_i + \mu^C(\partial_i u_j + \partial_j u_i) \ . \tag{14}$$

Substituting eq. (14) into eq. (12), we obtain the second-order derivative version of Carcione's first-order formulation [eqs. (B-1) to (B-6)]

$$-\rho\omega^2\tilde{\mathbf{u}} = \nabla[(\lambda^C + 2\mu^C)\nabla\cdot\tilde{\mathbf{u}}] - \nabla \times (\mu^C\nabla \times \tilde{\mathbf{u}}) \ , \tag{15}$$

where $\lambda^C + 2\mu^C$ is the complex P-wave modulus, and μ^C is the complex S-wave modulus.

Conditional equivalence of Robertsson's and Carcione's formulations

From eqs. (9) and (10), the complex moduli for Robertsson's formulation are

$$M_m^C = M_m\{(1/L)\sum_{l=1}^{L}[(1+i\omega\tau_{\varepsilon l}^m)/(1+i\omega\tau_{\sigma l})]\} \ , \text{ for } m = p,s, \tag{16}$$

where $M_p = \lambda + 2\mu$ and $M_s = \mu$, are the relaxed moduli for P- and S-waves. Eq. (16) is the same as the complex modulus formulation (obtained by Fourier transforming the stress relaxation function) in Carcione (2007). Setting $\tau_{\sigma l}^{(1)} = \tau_{\sigma l}^{(2)} = \tau_{\sigma l}$, and $L_1 = L_2 = L$, $M_p^C = \lambda^C + 2\mu^C$, and $M_s^C = \mu^C$, the second-order version [eq. (15)] of Carcione's first-order formulation is equivalent to the second-order version (equation 8) of Robertsson's first-order formulation. Under the assumption that $\tau_{\sigma l}^{(1)} = \tau_{\sigma l}^{(2)} = \tau_{\sigma l}$, and $L_1 = L_2 = L$, Robertsson's formulation (Appendix A) is equivalent to Carcione's formulation (Appendix B). The effective quality factors of the P- and S-waves can be obtained using $Q_m = Re(M_m^C)/Im(M_m^C)$, for m = p,s. Refer to Savage et al. (2010) for the relation between Q_p and Q_k.

THE GENERALIZED ROBERTSSON'S FORMULATION

Carcione's formulation is more general than Robertsson's formulation, since the former allows different stress relaxation times for each P and S-wave relaxation mechanism. For Robertsson's formulation, the assumption of $\tau_{\sigma l}^p = \tau_{\sigma l}^s = \tau_{\sigma l}$ is fine when multiple relaxation mechanisms are superimposed to approximate constant Q, using the τ method to estimate relaxation times, but not when using only one relaxation mechanism, or when the other methods (for example, the positivity preserving method (Blanc et al., 2016) are used to estimate relaxation times from Q, where the P- and S-stress relaxation times are Q-dependent. When only one relaxation mechanism is used to approximate constant Q around frequency ω_0, the equations for calculating the stress and strain relaxation times are (Carcione, 2007)

$$\tau_\sigma^m = Q_m/[1 + \sqrt{(1+Q_m^2)}]\omega_0 \quad , \tag{17}$$

and

$$\tau_\varepsilon^m = 1/\tau_\sigma^m \omega_0^2 \quad , \tag{18}$$

where Q_m, for m = p,s are the quality factors of P- and S-waves, and ω_0 is the dominant frequency of the source wavelet. Both the stress and strain relaxation times are functions of quality factor Q.

Generalization of Robertsson's formulation

The purpose of generalising Robertsson's formulation is to introduce different P- and S-wave stress relaxation times into the formulation, so it can be used when the P- and S-stress relaxation times are different in the input model.

The constitutive equation for the normal stress component (Robertsson et al., 1994) is

$$\sigma_{ij} = \dot{\Pi} * \nabla \cdot \mathbf{u} - 2\dot{M} * (\nabla\cdot\mathbf{u} - \partial_j u_i) \quad , \quad \text{for } i = j, \tag{19}$$

and for the shear stress component (Robertsson et al., 1994) is

$$\sigma_{ij} = \dot{M} * (\partial_j u_i + \partial_i u_j) \quad , \quad \text{for } i \neq j, \tag{20}$$

where we introduce different stress relaxation times, and define the relaxation functions for P- and S-waves as

$$\Pi = \pi\{1 + (1/L_p) \sum_{l=1}^{L_p} [(\tau^p_{\varepsilon l}/\tau^p_{\sigma l}) - 1] e^{-t/\tau^p_{\sigma l}}\} H(t) \quad , \tag{21}$$

$$M = \mu\{1 + (1/L_s) \sum_{l=1}^{L_s} [(\tau^s_{\varepsilon l}/\tau^s_{\sigma l}) - 1] e^{-t/\tau^s_{\sigma l}}\} H(t) \quad , \tag{22}$$

$$\pi = \lambda + 2\mu \quad , \tag{23}$$

and $\tau^m_{\sigma l}$, where m = p,s, are the stress relaxation times of the l-th mechanism for P- and S-waves, respectively. $*$ is the convolution operator, H(t) is the Heaviside function, and L_p and L_s are the numbers of P- and S-wave relaxation mechanisms, and thus can be set to be different or the same (Carcione, 1993). Usually in viscoelastic modeling, we set $L_p = L_s$, but this is not necessary.

To introduce different stress relaxation times for Q_P and Q_S, we define separate memory variables $r^{\Pi l}$ associated with Π, and r^{Ml}_{ij} for M, where i,j = x,y,z, by evaluating the convolution. Taking the time derivatives of eqs. (19) and (20), we obtain

$$\dot{\sigma}_{ij} = \pi^U \nabla\cdot\mathbf{v} - 2\mu^U(\nabla\cdot\mathbf{v} - \partial_j v_i) + (\sum_{l=1}^{L_p} r^{\Pi l} - \sum_{l=1}^{L_s} r^{Ml}_{ij}) \quad , \quad \text{for } i = j, \tag{24}$$

$$\dot{\sigma}_{ij} = \mu^U(\partial_j v_i + \partial_i v_j) + \sum_{l=1}^{L_s} r^{Ml}_{ij} \quad , \quad \text{for } i \neq j, \tag{25}$$

and

$$r^{\Pi l} = -(\pi/L)[(\tau^p_{\varepsilon l}/\tau^p_{\sigma l}) - 1](1/\tau^p_{\sigma l}) e^{-t/\tau^p_{\sigma l}} H(t) * \nabla\cdot\mathbf{v} \quad , \quad \text{for } l = 1,\ldots,L_p, \tag{26}$$

$$r^{Ml}_{ij} = -2(\mu/L)[(\tau^s_{\varepsilon l}/\tau^s_{\sigma l}) - 1](1/\tau^s_{\sigma l}) e^{-t/\tau^s_{\sigma l}} H(t) * (\nabla\cdot\mathbf{v} - \partial_j v_i) \quad , \quad \text{for } i = j, \text{ and } l = 1,\ldots,L_s, \tag{27}$$

and for the off-diagonal memory variable components,

$$r^{Ml}_{ij} = -(\mu/L)[(\tau^s_{\varepsilon l}/\tau^s_{\sigma l}) - 1](1/\tau^s_{\sigma l}) e^{-t/\tau^s_{\sigma l}} H(t) * (\partial_j v_i + \partial_i v_j) \quad , \quad \text{for } i \neq j, \text{ and } l = 1,\ldots,L_s. \tag{28}$$

Taking a time derivative, eqs. (26)-(28) become

$$\dot{r}^{\Pi l} = -(1/\tau^{p}_{\sigma l})r^{\Pi l} - (\pi/L)[(\tau^{p}_{\varepsilon l}/\tau^{p}_{\sigma l})-1](1/\tau^{p}_{\sigma l})\nabla\cdot\mathbf{v} \ , \quad \text{for } l=1,\ldots,L_p, \qquad (29)$$

$$\dot{r}^{Ml}_{ij} = -(1/\tau^{s}_{\sigma l})r^{Ml}_{ij} - 2(\mu/L)[(\tau^{s}_{\varepsilon l}/\tau^{s}_{\sigma l})-1](1/\tau^{s}_{\sigma l})(\nabla\cdot\mathbf{v} - \partial_j v_i) \ ,$$

$$\text{for } i = j, \text{ and } l = 1,\ldots,L_s, \qquad (30)$$

and

$$\dot{r}^{Ml}_{ij} = -(1/\tau^{s}_{\sigma l})r^{Ml}_{ij} - (\mu/L)[(\tau^{s}_{\varepsilon l}/\tau^{s}_{\sigma l})-1](1/\tau^{s}_{\sigma l})(\partial_j v_i + \partial_i v_j) \ ,$$

$$\text{for } i \neq j, \text{ and } l = 1,\ldots,L_s, \qquad (31)$$

where $\mathbf{v}$ is the particle velocity vector, and v_i, for $i = x,y,z$ are the particle velocity components. The equation of motion is

$$\rho\dot{v}_i = \partial_j\sigma_{ij} \ . \qquad (32)$$

With the introduction of separate memory variables related to viscoelastic relaxation functions (Π and M in the isotropic elastic case), instead of using composite memory variables for each stress component, we generalise Robertsson's formulation into a viscoelastic formulation with different stress relaxation times for P- and S-waves. We refer to eqs. (24), (25) and (29)-(32) as the generalized Robertsson's formulation. When $\tau^{p}_{\sigma l} = \tau^{s}_{\sigma l} = \tau_{\sigma l}$, and using the composite memory variable r^{l}_{ii} instead of r^{Ml}_{ij} and $r^{\Pi l}$, the generalized Robertsson's formulation reduces to Robertsson's formulation.

In 2D, the viscoelastic generalized Robertsson's equations are

$$\dot{v}_x = \rho^{-1}[(\partial\sigma_{xx}/\partial x) + (\partial\sigma_{xz}/\partial z)] \ , \qquad (33)$$

$$\dot{v}_z = \rho^{-1}[(\partial\sigma_{xz}/\partial x) + (\partial\sigma_{zz}/\partial z)] \ , \qquad (34)$$

$$\dot{\sigma}_{xx} = \pi^{U}\nabla\cdot\mathbf{v} - 2\mu^{U}(\partial v_z/\partial z) + (\sum_{l=1}^{L_p} r^{\Pi l} - \sum_{l=1}^{L_s} r^{Ml}_{xx}) \ , \qquad (35)$$

$$\dot{\sigma}_{zz} = \pi^{U}\nabla\cdot\mathbf{v} - 2\mu^{U}(\partial v_x/\partial x) + (\sum_{l=1}^{L_p} r^{\Pi l} - \sum_{l=1}^{L_s} r^{Ml}_{zz}) \ , \qquad (36)$$

$$\dot{\sigma}_{xz} = \mu^{U}[(\partial v_z/\partial x) + (\partial v_x/\partial z)] + \sum_{l=1}^{L_s} r^{Ml}_{xz} \ , \qquad (37)$$

$$\dot{r}^{\Pi l} = -(1/\tau^{p}_{\sigma l})r^{\Pi l} - (\pi/L)[(\tau^{p}_{\varepsilon l}/\tau^{p}_{\sigma l})-1](1/\tau^{p}_{\sigma l})\nabla\cdot\mathbf{v} \ , \quad \text{for } l=1,\ldots,L_p, \qquad (38)$$

$$\dot{r}^{Ml}_{xx} = -(1/\tau^{s}_{\sigma l})r^{Ml}_{xx} - 2(\mu/L)[(\tau^{s}_{\varepsilon l}/\tau^{s}_{\sigma l})-1](1/\tau^{s}_{\sigma l})(\partial v_z/\partial z) \ ,$$

$$\text{for } l = 1,\ldots,L_s, \qquad (39)$$

$$\dot{r}^{Ml}_{zz} = -(1/\tau^{s}_{\sigma l})r^{Ml}_{zz} - 2(\mu/L)[(\tau^{s}_{\varepsilon l}/\tau^{s}_{\sigma l})-1](1/\tau^{s}_{\sigma l})(\partial v_x/\partial x) ,$$

$$\text{for } l = 1,\ldots,L_s, \tag{40}$$

and

$$\dot{r}^{Ml}_{xz} = -(1/\tau^{s}_{\sigma l})r^{Ml}_{xz} - (\mu/L)[(\tau^{s}_{\varepsilon l}/\tau^{s}_{\sigma l})-1](1/\tau^{s}_{\sigma l})[(\partial v_z/\partial x) + (\partial v_x/\partial z)] ,$$

$$\text{for } l = 1,\ldots,L_s. \tag{41}$$

Equivalence of Carcione's and generalized Robertsson's formulations

It is straightforward to extend the derivation of the second-order derivative viscoelastic equation [eq. (8)] to obtain the second-order form of the generalized Robertsson's formulation. The second-order derivative generalized Robertsson's equation is

$$-\rho\omega^2\tilde{\mathbf{u}} = \nabla\{\pi(1/L_p)\sum_{l=1}^{L_p}[(1+i\omega\tau^{p}_{\varepsilon l})/(1+i\omega\tau^{p}_{\sigma l})]\nabla\cdot\tilde{\mathbf{u}}\}$$

$$- \nabla\times\{\mu(1/L_s)\sum_{l=1}^{L_s}[(1+i\omega\tau^{s}_{\varepsilon l})/(1+i\omega\tau^{s}_{\sigma l})]\nabla\times\tilde{\mathbf{u}}\} . \tag{42}$$

The complex moduli are

$$M^{C}_{m} = M_m\{(1/L_m)\sum_{l=1}^{L_m}[(1+i\omega\tau^{m}_{\varepsilon l})/(1+i\omega\tau^{m}_{\sigma l})]\} , \quad \text{for } m = p,s. \tag{43}$$

Note that, compared with eq. (16), eq. (43) allows individual stress relaxation times and numbers of relaxation mechanisms. We usually set the number of relaxation mechanisms of P- and S-waves to be equal. The generalized Robertsson's formulation is equivalent to Carcione's formulation without the limitation of having the same stress relaxation times for P- and S-waves.

NUMERICAL EXAMPLES

In this section, we first consider P- and S-wave stress relaxation times that are the same within each mechanism, and the strain relaxation times are calculated using the τ method (Blanch et al., 1995) with three mechanisms, from a given Q model. Then we consider different P- and S-wave stress relaxation times, by using a single relaxation mechanism, with P- and S-wave stress and strain relaxation times calculated from eqs. (17) and (18). All the numerical examples in the following sections are calculated using the first-order equations, eqs. (A-1) to (A-5) (for Robertsson's formulation), eqs. (B-1) to (B-6) (for

Carcione's formulation), and eqs. (24), (25) and (29)-(32) (for the generalized Robertsson's formulation).

A homogeneous isotropic viscoelastic model

We start with a simple homogeneous model, with the relaxed P-wave velocity $V_P = 3000$ m/s, S-wave velocity $V_S = V_P/1.7$, $Q_P = 40$, and $Q_S = 20$. The source is at (x, z) = (2.05, 2.03) km, and we use a composite explosive P-wave and shear S-wave source, with a Ricker wavelet of 25 Hz dominant frequency. The receivers are at depth z = 0.03 km.

Fig. 1 shows the X- and Z-components of particle velocity seismograms calculated by Carcione's and by Robertsson's formulations using three relaxation mechanisms, and the seismogram differences. The generalized Robertsson's formulation reduces to Robertsson's formulation when the P- and S-wave stress relaxation times are the same, and thus the seismograms for the generalized Robertsson's formulation are not shown. The first arrival is the direct P-wave, and the second arrival is the direct S-wave. The seismograms computed by Carcione's (Figs. 1a and 1d) and by Robertsson's (Figs. 1b and 1e) formulations are visually identical (Figs. 1c and 1f). When using multiple relaxation mechanisms, the P- and S-wave stress relaxation times share the same values

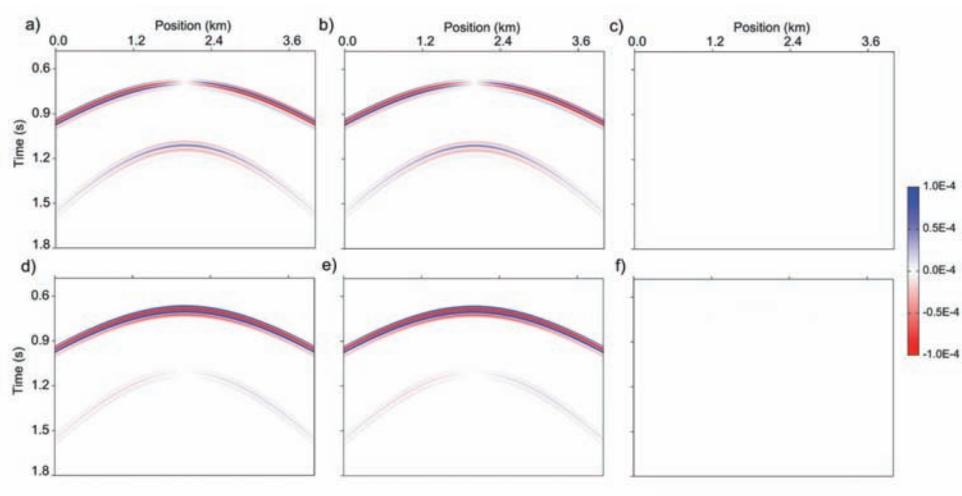

Fig. 1. X- (a-c) and Z- (d-f) components of particle velocity viscoelastic seismograms calculated by Carcione's and Robertsson's formulations, and their differences, from left to right, respectively. Three relaxation mechanisms (L = 3) are used, P- and S-stress relaxation times are the same. Relative amplitude is scaled the same in all panels.

(the reciprocals of the reference angular frequencies), which meets the conditions for the equivalence of Carcione's and Robertsson's formulations. Because Carcione's and Robertsson's formulations have different parameterizations of the P-wave attenuation, the corresponding effective Q values over frequency are not exactly the same; thus, after magnification of Figs. 1c and 1f (not shown here), small differences in the P-wave seismograms are detectable. However, as the S-wave attenuation parameterization is the same in both formulations, the S-waves match exactly.

Fig. 2 shows Q_P and Q_S measured from the seismograms in Fig. 1. We use the spectral ratio method (Båth, 1974; Kang and McMechan, 1994) to measure Q as a function of frequency. Q_P and Q_S from the seismograms calculated by Carcione's formulation overlap with those measured from the seismograms by Robertsson's formulation. The measured Q_P and Q_S approach the constant Q_P, and Q_S and the SLS approximations for frequencies ~ 6-50 Hz. The minor deviations of measured Q from the SLS approximation come from the measurement error in the spectral ratio method.

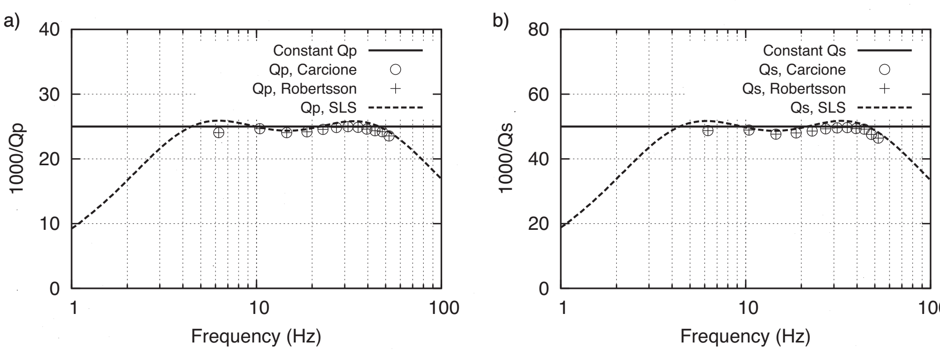

Fig. 2. (a) Q_P and (b) Q_S measured from the seismograms in Fig. 1. Solid line: constant Q; dashed line: SLS approximation with three relaxation mechanisms; o: Q measured from seismograms modelled by Carcione's formulation; +: Q measured from seismograms modelled by Robertsson's formulation.

When using one relaxation mechanism, seismograms modelled by Carcione's, Robertsson's, and the generalized Robertsson's formulations are shown in Fig. 3. Fig. 4 shows the differences between the seismograms calculated by the Carcione's and Robertsson's formulations, and between those of the Carcione's and the generalized Robertsson's formulations. The S-waves simulated by Robertsson's formulation (Figs. 3b and 3e) are different from those modelled by Carcione's (Figs. 3a and 3d) or the generalized Robertsson's formulation (Figs. 3c and 3f). Obvious seismogram differences are observed in Figs. 4a and 4c. There is no visible difference (Figs. 4b and 4d), between the seismograms calculated by Carcione's, and by the generalized Robertsson's, formulations.

With one relaxation mechanism to introduce P- and S-wave attenuation, respectively, the P- and S-wave stress relaxation times are not the same [eq.(17)]. To use Robertsson's formulation, we set the stress relaxation time equal to the P-wave value, which produces an error in the effective Q_S modelled during wavefield extrapolation, and thus affects the S-waveforms (Figs. 4a and 4c). Fig. 5 shows the measured Q_P and Q_S from seismograms of the three formulations. For Robertsson's formulation, there is an obvious deviation of the measured Q_S (the symbol +) from the SLS approximation (the dashed line) in Fig. 5b, while the measured Q_P (the symbol +) matches the SLS approximation well in Fig. 5, and approximates the constant Q around the dominant frequencies. If we set the stress relaxation times equal to the corresponding S-wave value, the direct S-wave will be correctly modelled, and error will be introduced into the P-wave attenuation. Both Carcione's and the generalized Robertsson's formulations allow different values of P- and S-wave stress relaxation times, which makes the Q_P and Q_S correctly modelled in the seismograms, and the measured Q_P and Q_S are consistent with the SLS approximation. Both $1000/Q_P$ and $1000/Q_S$ associated with Carcione's and the

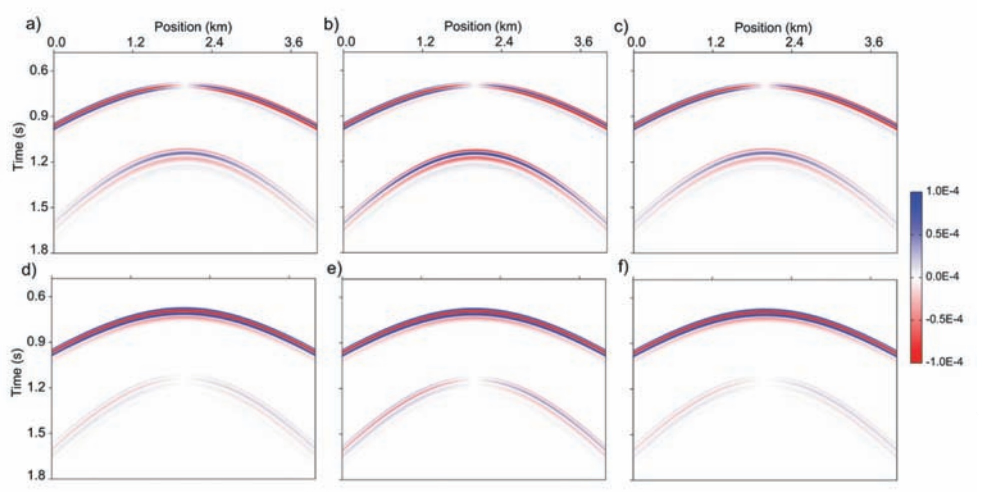

Fig. 3. X- (a-c) and Z- (d-f) component viscoelastic seismograms calculated by Carcione's, Robertsson's, and the generalized Robertsson's formulations, from left to right, respectively. One relaxation mechanism (L = 1) is used, P- and S-stress relaxation times are different. Relative amplitude is scaled the same in all panels.

generalized Robertsson's formulations decrease away from the dominant frequency; the approximation to constant Q using one relaxation mechanism is not as good as when using three relaxation mechanisms (Fig. 2), as expected (Blanch et al., 1995), However, a single relaxation can still provide a reasonable approximation to constant Q at the dominant frequency, if the seismic data have a narrow bandwidth (Blanch et al., 1995).

An isotropic viscoelastic version of the Hess VTI model

The second numerical example uses the vertical P-wave velocity and density of the 2D Hess VTI model as the relaxed P-wave velocity and density in the isotropic viscoelastic model. We resample the model by keeping every third grid point, and set the grid increment to 6.0 m. The relaxed S-wave velocity $V_S = V_P/1.7$. When $V_P < 4500$ m/s, we set $Q_P = 100 \times V_P/(1524$ m/s), and when $V_P \geq 4500$ m/s, we set $Q_P = 800$. $Q_S = Q_P/2$. The resulting V_P, density, and the constructed Q_P models are shown in Fig. 6. We use an explosive P-wave source with a Ricker wavelet of 25 Hz dominant frequency. The time step is 5.0E-4 s, and the total recording time is 3.0 s. The source is at (x, z) = (3.0, 0.03) km, and the receivers are at depth z = 0.03 km.

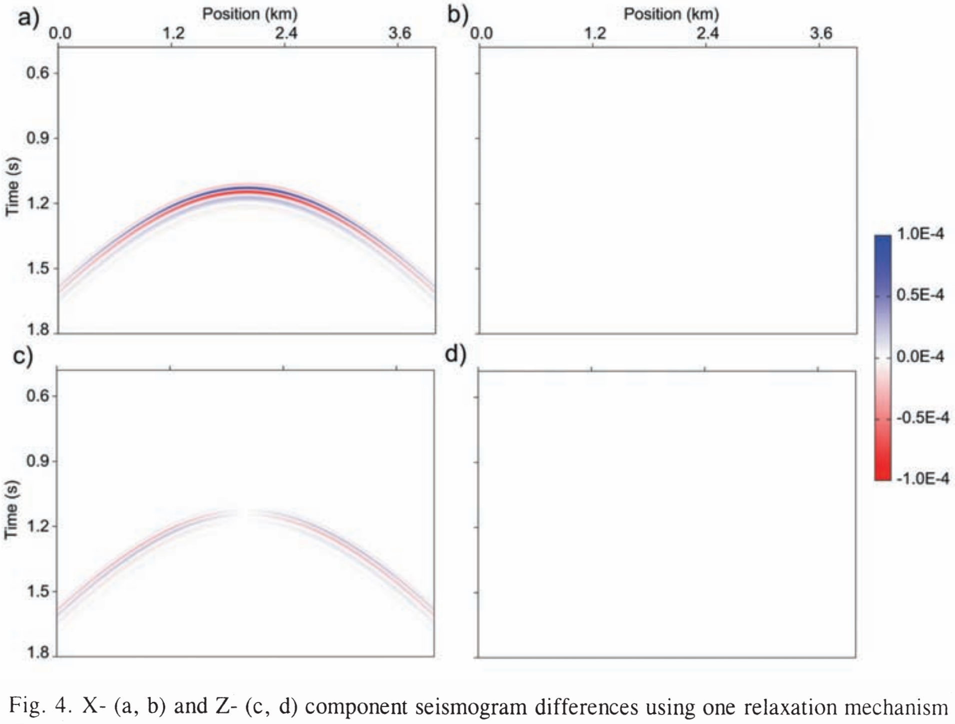

Fig. 4. X- (a, b) and Z- (c, d) component seismogram differences using one relaxation mechanism (L = 1), P- and S-stress relaxation times are different; (a, c) seismogram differences between Carcione's and Robertsson's formulations; (b, d) seismogram differences between Carcione's and the generalized Robertsson's formulations. Relative amplitude is scaled the same as in Fig. 3.

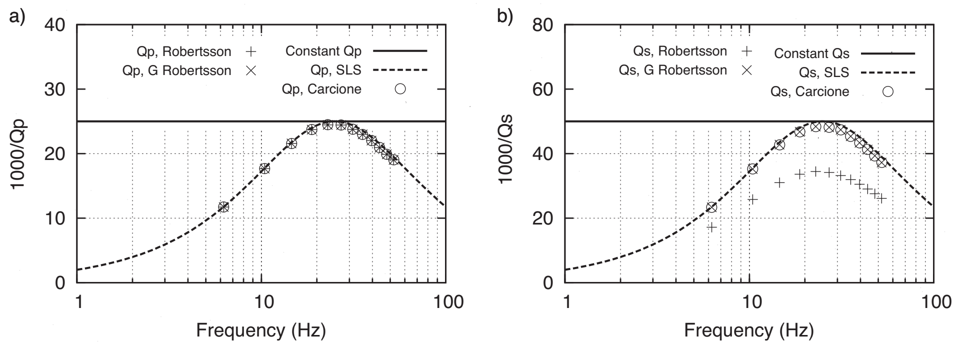

Fig. 5. (a) Q_P and (b) Q_S measured from seismograms in Fig. 3. Solid line: constant Q; dashed line: SLS approximation with one relaxation mechanism; o: Q measured from seismograms modelled by Carcione's formulation; +: Q measured from seismograms modelled by Robertsson's formulation; ×: Q measured from seismograms modelled by the generalized Robertsson's formulation. 'G' in the legends denotes 'generalized'.

Fig. 7 shows the X- and Z-component seismograms without attenuation, calculated by first-order elastic velocity-stress equations (Virieux, 1986). Fig. 8 shows the seismograms calculated using Carcione's and Robertsson's formulations with three relaxation mechanisms, and their differences. When compared with elastic seismograms without attenuation (Fig. 7), there is an obvious amplitude decrease when attenuation is included (Fig. 8). Fig. 9 shows X and Z component seismic traces at the representative position 2.4 km in Fig. 8. There is no visible difference between the seismograms computed by Carcione's and Robertsson's formulations (Figs. 8 and 9), which are equivalent when the P- and S-wave stress relaxation times are the same.

Fig. 10 shows the X- and Z-component seismograms using one relaxation mechanism, using Carcione's (Figs. 10a and 10d), Robertsson's (Figs. 10b and 10e), and the generalized Robertsson's (Figs. 10c and 10f) formulations. Fig. 11 shows the seismogram differences. Representative X- and Z-component seismic traces from position 2.4 km are shown in Fig. 12. Obvious differences are observed for the X- (Figs. 11a and 11c) and Z-component (Figs. 12c and 12d) seismograms of Carcione's and Robertsson's formulations using one relaxation mechanism, obtained by setting the stress relaxation time in Robertsson's formulation equal to the P-wave stress relaxation time. Q_S is not correctly incorporated into Robertsson's formulation. The difference in Q_S will influence both the P- and S-wave reflection and transmission coefficients (Sidler et al., 2008), and thus there are differences in both reflected P- and S-waves in Fig. 11. The seismograms are visually identical for Carcione's and the generalized Robertsson's formulations (compare Figs. 11b and 11d, and Figs. 12a and 12b).

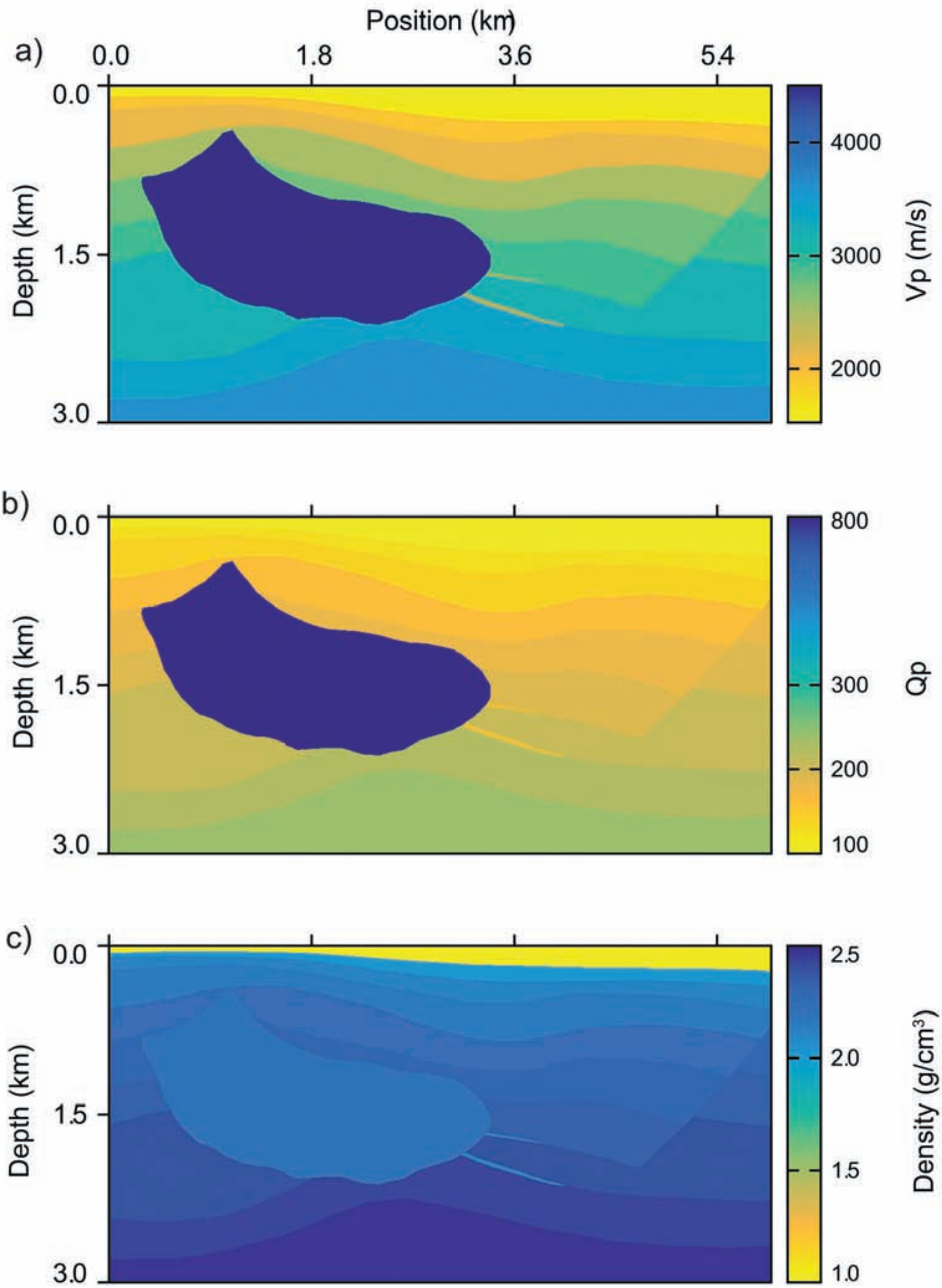

Fig. 6. (a) P-wave velocity; (b) P-wave quality factor Q_P; (c) density of the modified Hess model, $V_S = V_P/1.7$, and $Q_S = Q_P/2$.

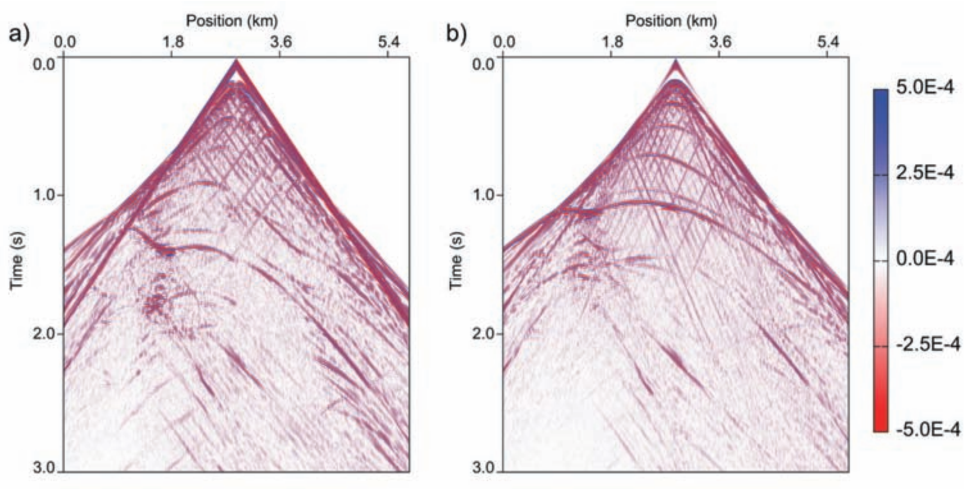

Fig. 7. X- (a) and Z- (b) component elastic seismograms without attenuation.

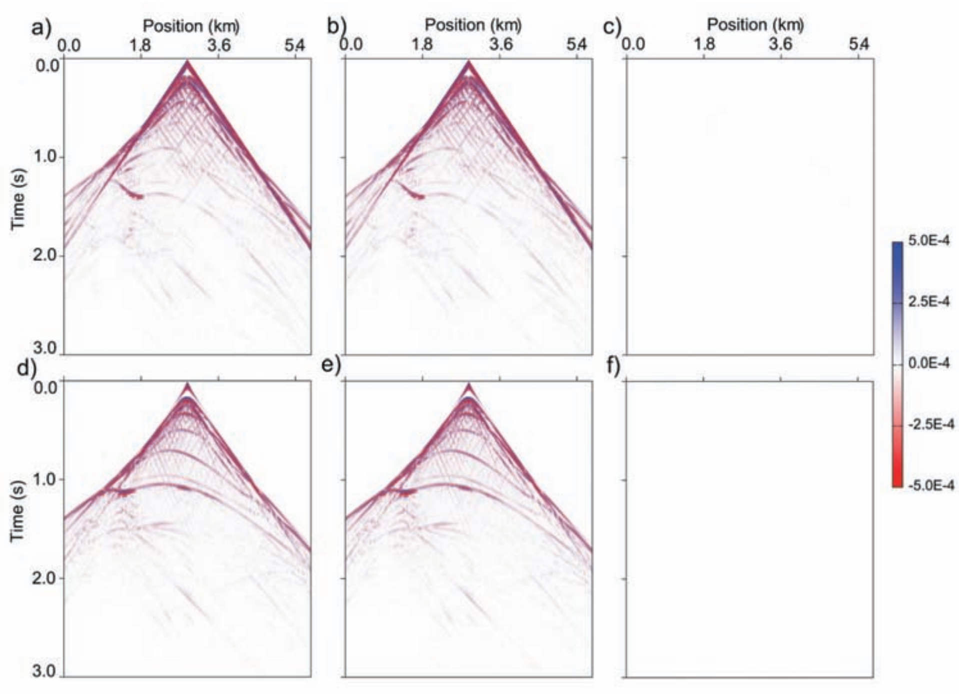

Fig. 8. X- (a-c) and Z- (d-f) component viscoelastic seismograms calculated by Carcione's and (the generalized) Robertsson's formulations, and their differences, from left to right, respectively. Three relaxation mechanisms (L = 3) are used, P- and S-stress relaxation times are the same. Relative amplitude is scaled the same.

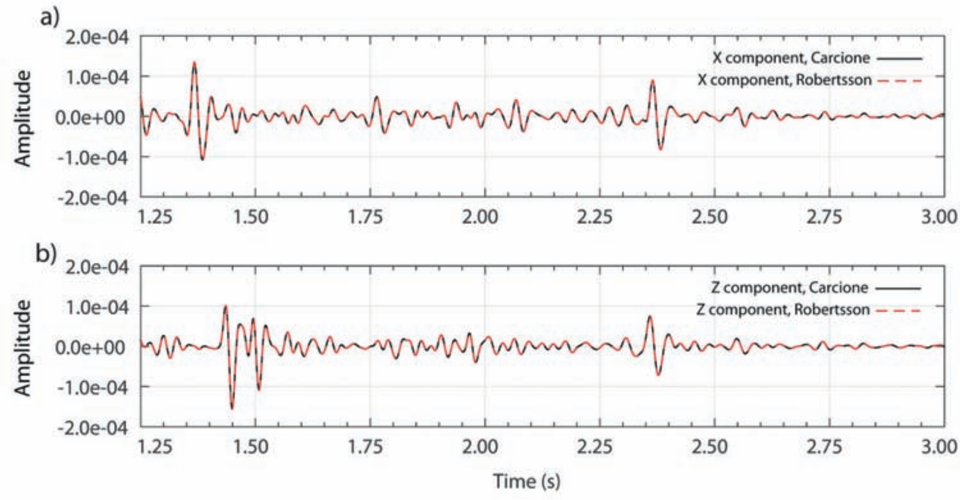

Fig. 9. X- (a) and Z- (b) component seismic traces for the position 2.4 km of Fig. 8.

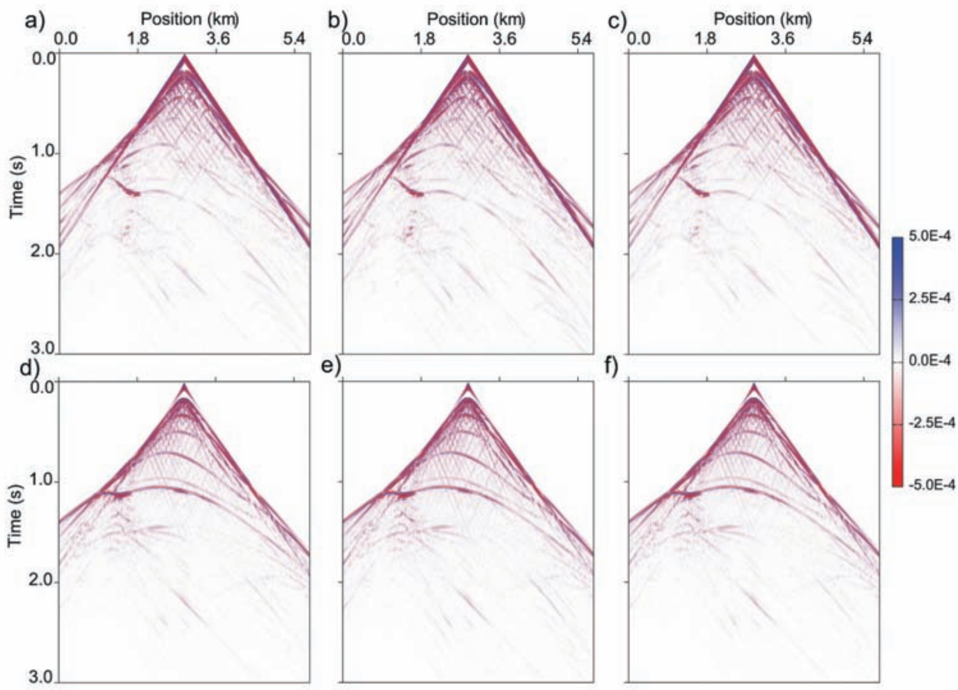

Fig. 10. X- (a-c) and Z- (d-f) components of viscoelastic seismograms calculated by Carcione's, Robertsson's, and the generalized Robertsson's formulations, from left to right, respectively. One relaxation mechanism (L = 1) is used, P- and S-stress relaxation times are different. Relative amplitude is scaled the same in all panels, and are the same as in Fig. 8.

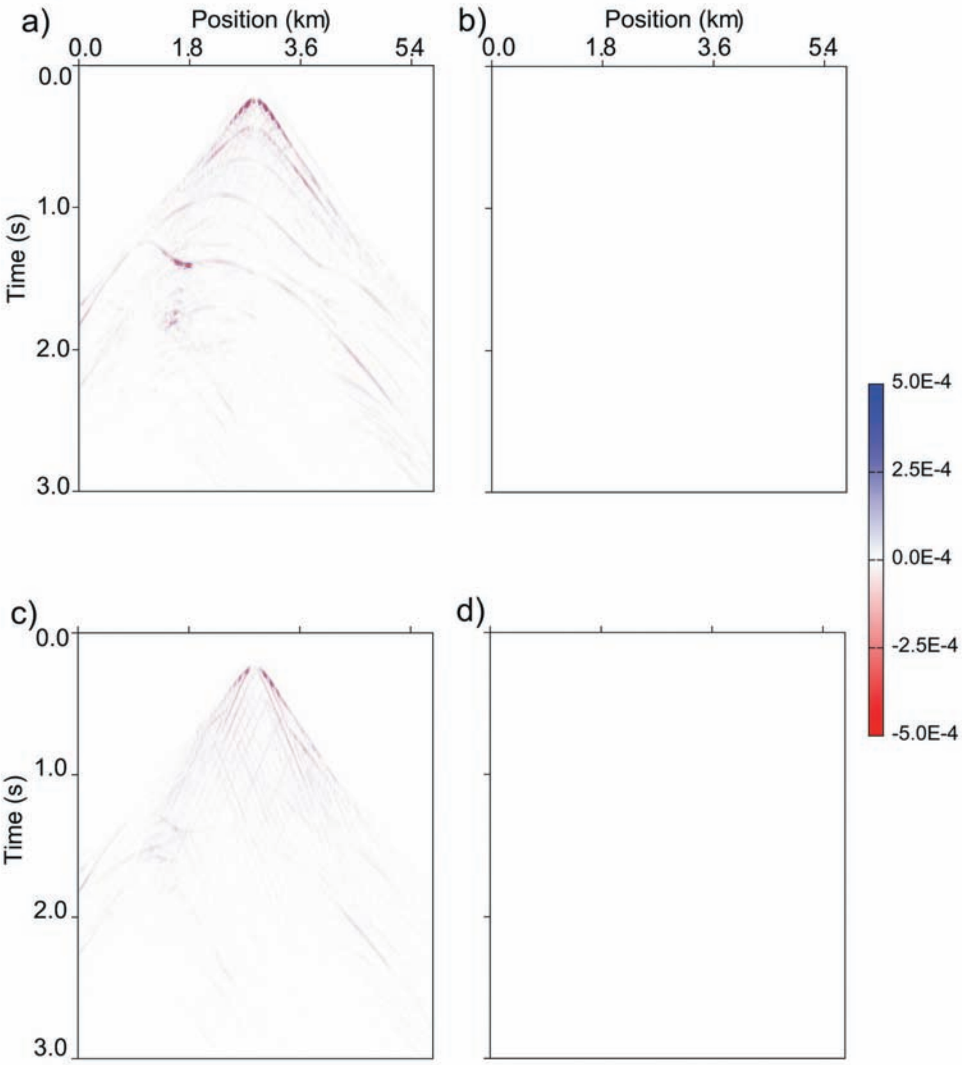

Fig. 11. X- (a, b) and Z- (c, d) components of seismogram differences using one relaxation mechanism (L = 1), P- and S-stress relaxation times are different; (a, c) seismogram differences between Carcione's and Robertsson's formulations; (b, d) seismogram differences between Carcione's and the generalized Robertsson's formulations. Relative amplitude is scaled the same as in Fig. 8.

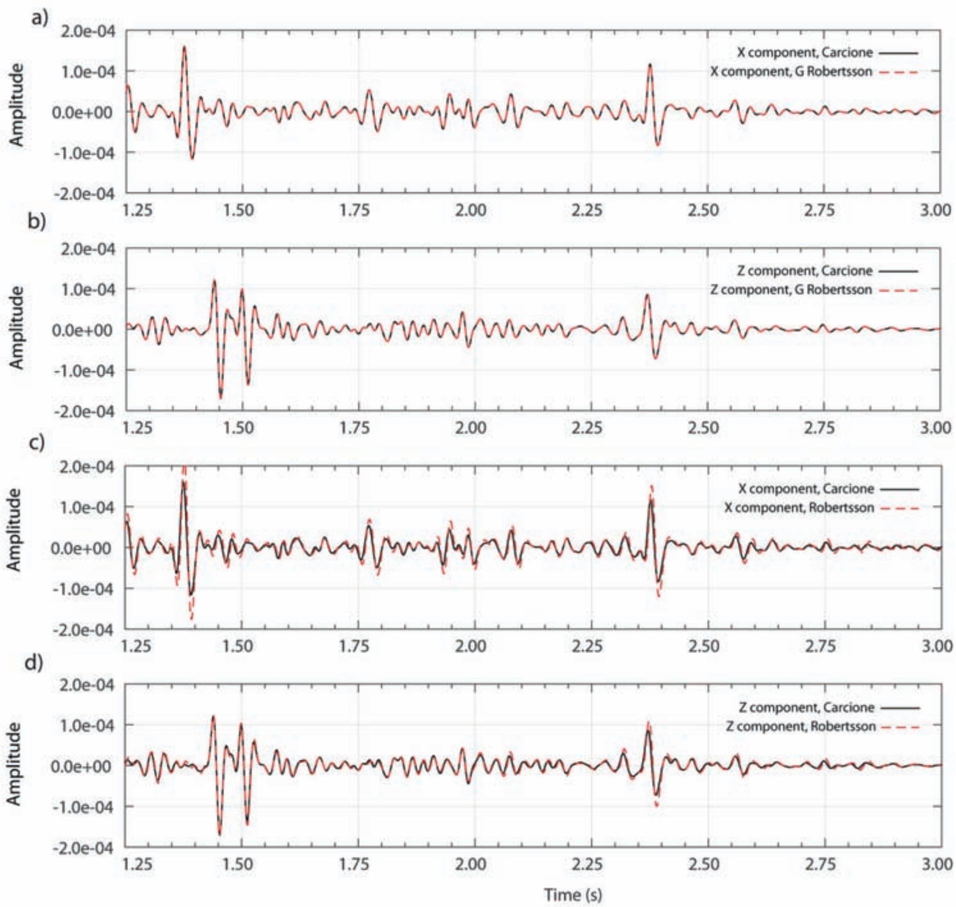

Fig. 12. X- (a, c) and Z- (b, d) component seismic traces for the position 2.4 km of Fig. 10. (a, b) seismic traces of Carcione's and the generalized Robertsson's formulations for one relaxation mechanism (L = 1). (c, d) seismic traces of Carcione's and Robertsson's formulations.

Now consider the computational cost, of viscoelastic modeling by Carcione's, Robertsson's, and the generalized Robertsson's formulations, in Table 1; with the number of model grid points 1000 in the X-, and 520 in the Z-direction and 6000 time steps, the three formulations have comparable cost. Although there is one more memory variable in the generalized Robertsson's formulation than in the traditional Robertsson's formulation, the corresponding increase in computational cost is offset because, compared with eq. (A-4), there is less computational cost in each of the eqs. (29) and (30).

Table 1. Computational time (wall clock time multiplied by the number of threads used) for Carcione's, Robertsson's, and the generalized Robertsson's formulations. For comparison, for the elastic modeling, L = 0, and the computational time is 1548 s.

Equation	Carcione	Robertsson	G Robertsson
Computation time (s) L = 1	2136	2016	2076
Computation time (s) L = 3	2880	2692	2796

DISCUSSION

Extension to general anisotropic viscoelastic media

The generalized Robertsson's formulation can be extended to general anisotropic viscoelastic media [for anisotropic attenuation with the same geometry as the anisotropic velocity (Zhu and Tsvankin, 2006)]. With Einstein notation, the viscoelastic constitutive stress-strain relation in general anisotropic viscoelastic media is (Christensen, 1982; Hestholm, 1999; Ruud and Hestholm, 2005).

$$\sigma_{ij} = \dot{C}_{ijmn} * \epsilon_{mn} \quad , \tag{44}$$

and

$$C_{ijmn}(t) = C^R_{ijmn}\{1 + (1/L)\sum_{l=1}^{L} [(\tau_{\varepsilon l}^{ijmn}/\tau_{\sigma l}^{ijmn}) - 1]e^{-t/\tau_{\sigma l}^{ijmn}}\}H(t) \quad , \tag{45}$$

where σ_{ij} and ϵ_{mn} are stress and strain components, respectively, and $C_{ijmn}(t)$ is a fourth-order time-dependent tensor called the relaxation function (Carcione et al., 1988c; Rudd and Hestholm, 2005). C^R_{ijmn} is the relaxed modulus, $\tau_{\varepsilon l}^{ijmn}$ and $\tau_{\sigma l}^{ijmn}$ are the l-th strain and stress relaxation times for the Q tensor. Applying a time derivative to eq. (44), we obtain

$$\dot{\sigma}_{ij} = C^U_{ijmn}\dot{\epsilon}_{mn} + \sum_{l=1}^{L} r^l_{ijmn} \quad , \tag{46}$$

where C^U_{ijmn} is the unrelaxed modulus, and

$$r^l_{ijmn} = -(C^R_{ijmn}/L)[(\tau_{\varepsilon l}^{ijmn}/\tau_{\sigma l}^{ijmn}) - 1](1/\tau_{\sigma l}^{ijmn})e^{-t/\tau_{\sigma l}^{ijmn}} H(t) * \dot{\epsilon}_{mn} \quad , \quad \text{for } l = 1,\ldots,L. \tag{47}$$

Applying a time derivative to eq. (47),

$$\dot{r}^l_{ijmn} = -(1/\tau_{\sigma l}^{ijmn})r^l_{ijmn} - (1/\tau_{\sigma l}^{ijmn})(C^R_{ijmn}/L)[(\tau_{\varepsilon l}^{ijmn}/\tau_{\sigma l}^{ijmn}) - 1]\dot{\epsilon}_{mn} \quad , \quad \text{for } l = 1,\ldots,L. \tag{48}$$

Ruud and Hestholm (2005) derived similar anisotropic viscoelastic equations, but with composite memory variables for each stress component, which requires less memory than our approach, but with the limitation that the same stress relaxation times have to be used for all Q values.

In the numerical examples, when using multiple relaxation mechanisms for P- and S-waves, the stress relaxation times of P- and S-waves are the same; both are the reciprocals of the selected angular frequencies, as $\tau_{\sigma l} = 1/\omega_l$ (Blanch et al., 1995). The P- and S-wave stress relaxation times can also be different, when using other methods such as the positivity preserving method (Blanc et al., 1995) to estimate relaxation times from Q. The fact that the P- and S-wave stress relaxation times can be different for one relaxation mechanism, or multiple relaxation mechanisms in the positivity preserving method, makes the inclusion of different P and S-wave stress relaxation times in the viscoelastic propagators necessary to obtain physically consistent results.

On the 1/L in the relaxation function of viscoelastic media

The generalized standard linear solid (GSLS) model has been widely used for incorporating intrinsic attenuation into seismic modeling. However, apparent inconsistencies have been observed in the definitions of the GSLS model; 1/L is included, or not included, in the viscoelastic modulus and relaxation functions. Later publications (Mozco and Kristek, 2005; Carcione, 2007) indicate that there is a missing of 1/L in the GSLS model when Liu et al. (1976) generalized the standard linear solid (L = 1) model to include more than one relaxation mechanisms (L ≥ 1, the GSLS model), and thus suggest that an error is introduced by omission of 1/L. However, here we prove that both definitions, with and without 1/L, are correct. The formulation difference is caused by different definitions of the strain relaxation times.

For the stress-strain relation in a viscoelastic medium based on the GSLS,

$$\sigma = \dot{C} * \epsilon \ , \tag{49}$$

the relaxation function C(t) is defined [with 1/L, as by Moczo and Kristeki (2005), Carcione (2007), Moczo et al. (2014), Yang et al. (2015), and Blanc et al. (2016)], as

$$C(t) = C^R\{1 + (1/L)\sum_{l=1}^{L} [(\tau_{\varepsilon l}/\tau_{\sigma l}) - 1]e^{-t/\tau_{\sigma l}}\}H(t) \ , \tag{50}$$

or [without 1/L, as by Carcione (1993), Robertsson et al. (1994), Blanch et al. (1995), Xu and McMechan (1995), Hestholm (1999) and Hestholm (2002)], as

$$C(t) = C^R\{1 + \sum_{l=1}^{L} [(\tau'_{\varepsilon l}/\tau'_{\sigma l}) - 1]e^{-t/\tau'_{\sigma l}}\}H(t) \quad . \tag{51}$$

Note that, to differentiate between the relaxation times in eqs. (50) and (51), we use $\tau'_{\varepsilon l}$ and $\tau'_{\sigma l}$ for strain and stress relaxation times in eq. (51). In the viscoelastic formulations of the previous sections, we adopt the definition of relaxation function in eq. (50). Now we prove that these two definitions are equivalent.

Fourier transforming eq. (50) and applying some algebraic operations,

$$\begin{aligned} M_c(\omega) &= (M^R/L) \sum_{l=1}^{L} [(1+i\omega\tau_{\varepsilon l})/(1+i\omega\tau_{\sigma l})] \\ &= M^R \sum_{l=1}^{L} \{[(1+i\omega\tau_{\varepsilon l})/L]/(1+i\omega\tau_{\sigma l})\} \quad . \end{aligned} \tag{52}$$

Similarly with eq. (51), we obtain

$$\begin{aligned} M_c(\omega)' &= M^R\{1 - L + \sum_{l=1}^{L} [(1+i\omega\tau'_{\varepsilon l})/(1+i\omega\tau'_{\sigma l})] \\ &= M^R \sum_{l=1}^{L} \{[i\omega(\tau'_{\varepsilon l}-\tau'_{\sigma l}) + (1+i\omega\tau'_{\sigma l})/L]/(1+i\omega\tau'_{\sigma l})\} \quad . \end{aligned} \tag{53}$$

Setting $M_c(\omega) = M_c(\omega)'$, and $\tau_{\sigma l} = \tau'_{\sigma l}$, we have

$$\tau_{\varepsilon l} - \tau'_{\sigma l} = L(\tau'_{\varepsilon l}-\tau'_{\sigma l}) \quad . \tag{54}$$

Thus we have proved that these two definitions of relaxation function in eqs. (50) and (51) are equivalent; both of them are correct, and the difference in appearance comes from different definitions of strain relaxation times. In other words, the formulations for estimating relaxation times from a given Q model has to be consistent with the definition of the relaxation function in the stress-strain relations for the wavefield modeling.

CONCLUSIONS

We derive the second-order displacement equations for viscoelastic media based on the GSLS, and use it to demonstrate that, with the same stress relaxation times for P- and S-waves, Robertsson's formulation is equivalent to

Carcione's formulation. Robertsson's formulation is generalized to include different P- and S-wave stress relaxation times, which improves the physical consistency of Q_P and Q_S modelled in the seismograms. The generalized Robertsson's formulation is equivalent to Carcione's formulation. With regard to computational efficiency, the three formulations have comparable cost. Considering both the accuracy and computational cost of intrinsic attenuation, the generalized Robertsson's and Carcione's formulation are recommended, since they contain the flexibility of different P- and S-wave stress relaxation times, and have almost the same computational cost as Robertsson's formulation. The main difference between the generalized Robertsson's and Carcione's formulation is in the parameterization. P-wave attenuation is directly introduced by the P-wave quality factor (Q_P) in the generalized Robertsson's formulation, and indirectly introduced by the bulk quality factor (Q_κ) in Carcione's formulation.

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APPENDIX A

VISCOELASTIC FORMULATION BASED ON THE GENERALIZED STANDARD LINEAR SOLID (SGLS): ROBERTSSON'S APPROACH

The first-order velocity, stress and memory variable equations are (Robertsson et al. 1994)

$$\rho \dot{v}_i = \partial_j \sigma_{ij} \ . \tag{A-1}$$

$$\dot{\sigma}_{ij} = \pi^U \nabla\cdot\mathbf{v} - 2\mu^U(\nabla\cdot\mathbf{v} - \partial_j v_i) + \sum_{l=1}^{L} r_{ij}^l \ , \quad \text{for } i = j, \tag{A-2}$$

$$\dot{\sigma}_{ij} = \mu^U(\partial_j v_i + \partial_i v_j) + \sum_{l=1}^{L} r_{ij}^l \ , \quad \text{for } i \neq j, \tag{A-3}$$

$$\dot{r}_{ij}^l = -(1/\tau_{\sigma l}) r_{ij}^l - (1/\tau_{\sigma l})\{(\pi/L)[(\tau_{\varepsilon l}^p/\tau_{\sigma l}) - 1]\nabla\cdot\mathbf{v} - 2(\mu/L)[(\tau_{\varepsilon l}^s/\tau_{\sigma l}) - 1](\nabla\cdot\mathbf{v} - \partial_j v_i)\} \ ,$$

$$\text{for } i = j \text{ and } l = 1,\ldots,L, \tag{A-4}$$

and

$$\dot{r}_{ij}^l = -(1/\tau_{\sigma l}) r_{ij}^l - (1/\tau_{\sigma l})(\mu/L)[(\tau_{\varepsilon l}^s/\tau_{\sigma l}) - 1](\partial_j v_i + \partial_i v_j) \ ,$$

$$\text{for } i \neq j \text{ and } l = 1,\ldots,L, \tag{A-5}$$

where $\mathbf{v}$ is the particle velocity vector; v_i, for i = x,y,z, are particle velocity components; σ_{ij} and r_{ij}^l, for i,j = x,y,z are the stress components and memory variables, respectively, for the l-th relaxation mechanism. $\pi^U = (\lambda + 2\mu)M^{Up}$, and $\mu^U = \mu M^{Us}$ are the unrelaxed P- and S-wave moduli, where $M^{Up} = 1 + (1/L)\Sigma_{l=1}^{L}[(\tau_{\varepsilon l}^p/\tau_{\sigma l}) - 1]$, $M^{Us} = 1 + (1/L)\Sigma_{l=1}^{L}[(\tau_{\varepsilon l}^s/\tau_{\sigma l}) - 1]$, and $\tau_{\varepsilon l}^p$ and $\tau_{\varepsilon l}^s$ are the strain relaxation times of the l-th relaxation mechanism for P-waves (Q_P) and S-waves (Q_S), respectively. $\tau_{\sigma l}$ is the stress relaxation time for the l-th relaxation mechanism. L is the number of relaxation mechanisms used in the GSLS model. The stress relaxation times for P- and S-waves are assumed to be the same. Note that, throughout this paper, we use the relaxation function with the 1/L factor. Compared with the original equations in Robertsson et al. (1994) (which used the relaxation function without 1/L), the 1/L factor is added into the memory variable equations and unrelaxed moduli. Please refer to the discussion for more details.

APPENDIX B

VISCOELASTIC FORMULATION BASED ON THE GENERALIZED STANDARD LINEAR SOLID (SGLS): CARCIONE'S APPROACH

The first-order velocity, stress and memory variable equations are (Carcione, 1993; Xu and McMechan, 1995)

$$\rho \dot{v}_i = \partial_j \sigma_{ij} \quad . \tag{B-1}$$

$$\dot{\sigma}_{ij} = (\lambda^U + 2\mu^U)\partial_j v_i + \lambda^U(\nabla\cdot\mathbf{v} - \partial_j v_i) + [\lambda + (2/D)\mu \sum_{l=1}^{\overline{L_1}} r_1^l + 2\mu \sum_{l=1}^{\overline{L_2}} r_{ij}^l \ , \quad \text{for } i = j, \tag{B-2}$$

$$\dot{\sigma}_{ij} = \mu^U(\partial_j v_i + \partial_i v_j) + \mu \sum_{l=1}^{\overline{L_2}} r_{ij}^l \ , \quad \text{for } i \neq j, \tag{B-3}$$

$$\dot{r}_1^l = \Theta(\phi_l^{(1)}/L_1) - (r_1^l/\tau_{\sigma l}^{(1)}) \ , \quad \text{for } l = 1,\ldots,L_1, \tag{B-4}$$

$$\dot{r}_{ij}^l = [\partial_j v_i - (\Theta/D)](\phi_l^{(2)}/L_2) - (r_{ij}^l/\tau_{\sigma l}^{(2)}) \ , \quad \text{for } i = j \text{ and } l = 1,\ldots,L_2, \tag{B-5}$$

and

$$\dot{r}_{ij}^l = (\partial_j v_i + \partial_j v_i)(\phi_l^{(2)}/L_2) - (r_{ij}^l/\tau_{\sigma l}^{(2)}) \ , \quad \text{for } l = 1,\ldots,L_2, \tag{B-6}$$

where v_i, for i = x,y,z, is the i-component of the vector particle velocities, σ_{ij}, for i,j = x,y,z is the stress component, r_{ij}^l, for i,j = 1,2,3 and r_1^l are memory variables for the l-th relaxation mechanism. $\Theta = (\partial v_x/\partial x) + (\partial v_y/\partial y) + (\partial v_z/\partial z)$. L_1 and L_2 are the numbers of relaxation mechanisms for P- and S-waves. ρ is density, $\lambda^U = (\lambda + 2\mu)M^{U1} - (2/D)\mu)M^{U2}$, and $\mu^U = \mu M^{U2}$, where λ and μ are relaxed Lamé constants, $M^{Uk} = 1 + (1/L_k)\Sigma_{l=1}^{L_k}[(\tau_{\varepsilon l}^{(k)}/\tau_{\sigma l}^{(k)}) - 1]$, and $\phi_l^{(k)} = [1 - (\tau_{\varepsilon l}^{(k)}/\tau_{\sigma l}^{(k)})]/\tau_{\sigma l}^{(k)})$, for k = 1,2. $\tau_{\varepsilon l}^{(k)}$ and $\tau_{\sigma l}^{(k)}$ are the strain and stress relaxation times for the P-wave (k = 1), and the S-wave (k = 2), the letter 'U' refers to 'unrelaxed', and D is the spatial dimension. Note that $\tau_{\varepsilon l}^{(1)}$ and $\tau_{\sigma l}^{(1)}$ will also affect S-waves, since they are calculated from the bulk quality factor Q_κ; following Carcione (1993), we call these the P-wave strain and stress relaxation times. Note that, throughout this study, we use the relaxation function with the 1/L factor. Compared with the original equations in Carcione (1993) and Xu and McMechan (1995) (which used the relaxation function without 1/L), the 1/L factor is added into the memory variable equations and unrelaxed moduli. Please refer to the discussion for more details.