A NUMERICAL MODELING OF WAVE PROPAGATION THAT IS INDEPENDENT OF COORDINATE TRANSFORMATION

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ABSTRACT

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It is a remarkable fact that Maxwell's equations under any coordinate transformation can be written in an identical mathematical form as the ones in Cartesian coordinates. However, in some particular coordinate transformations, like the cylindrical coordinate transformations, the physical properties becomes anisotropic, even if they are isotropic in the Cartesian coordinates. Even the permittivity can be anisotropic. We here review these fundamental results. The remarkable invariance of Maxwell's equations under coordinate transformation can also extend to elastodynamic wave equations by rewriting them in a new form. We have used this new form of the elastodynamic wave equations to describe a numerical solution of elastic wave propagation which is independent of coordinate transformation.

KEY WORDS: elastodynamic equations, Maxwell's equations, curvilinear coordinates, natural coordinates, physical coordinates, finite-difference solution.

INTRODUCTION

Now that the controlled-source electromagnetic (CSEM) acquisition technique has taken hold as an oil and gas exploration and production tool, there is a need to develop modeling and inversion methods to analyze CSEM data, and even to revamp classical petroleum seismology classes to include electromagnetic methods. These developments will greatly benefit from the significant progress made in the last four decades in seismic modeling and inversion and in the centuries of electromagnetic-wave studies. One important

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aspect of these developments is an understanding of the similarities of and differences between Maxwell's equations and elastic field equations. In Ikelle (2012) we describe examples of these equivalences in Cartesian coordinates. The seismology studies are not limited to Cartesian coordinates. For example, in the study of sonic logging and of earthquake sources, we often considered the wave propagation in cylindrical coordinates and even spherical coordinates. We here examine the similarities in and differences between Maxwell's equations and elastic field equations for other coordinate systems. Our formulation is quite general and is valid for any transformation of Cartesian coordinate systems, including transformation from Cartesian coordinates to curvilinear coordinates. The resulting formulas can be used to design numerical methods for simulating data which are independent of coordinate systems. In other words, they provide simple ways of applying numerical solutions designed for Cartesian coordinates, for example, to other coordinate systems. Our basic derivation of coordinate transformation is similar to Pendry et al. (2006), Milton et al. (2006), Yan et al. (2008), and other groups involved in the metamaterial sciences, as we discovered after our derivations.

In the Cartesian coordinate system, elastic wave propagation responses of subsurface models containing a curved air-solid interface are known to produce artifacts, especially diffractions, from the staircase discretization of this interface. Fornberg (1988) proposed a pseudospectral finite-difference solution to this problem by using a curved grid whose lines coincide with this interface. His method consists of solving wave equations in Cartesian coordinates. It involves first computing the spatial derivatives in the curved grid and then applying the chain rule to calculate the required spatial derivatives in the Cartesian coordinates. Many authors, including Tessmer el al. (1992), Carcione (1994), Nielsen et al. (1994), Hestholm and Ruud (1994), and Tessmer and Kosloff (1994), have extended his original solution for working in the space-time domain and with more-complex subsurface models. Komatitsh et al. (1996), and later Friis et al. (2001), have proposed alternative solutions to this problem by formulating wave equations directly in the curved grid through the use of covariant derivatives. Although the objectives in these papers are different from the objective of this paper, there are a number of similarities to our work in regard to the fact that we all use, in one form or another, the notion of covariant derivatives. These similarities are even explicit with regard to the work of Komatitsh et al. (1996) because they directly use the mathematics of contravariants and covariants. However, none of these authors realized that these mathematics can lead to new effective elastic parameters and to a new form of elastic wave equations. Yet these two observations are the cornerstones of this paper and they are also the reason why final formulas and results here are different from those described in these papers.

The remainder of the paper is divided into five sections. In the next section, we will review some of the basic formulas of coordinate transformations that we will need in our later derivations. In the third section, we review the results of the invariance of Maxwell's equations with coordinate systems. In the fourth section, we discuss the invariance of elastic wave equations with coordinate systems. One of the main results discussed in this section is that we can rewrite the equations of elastic wave propagation in a form for which they become coordinate invariant just like Maxwell's equations. In the fifth section, we use this form of elastic wave equations to propose a numerical solution for simulating synthetic data which is independent of coordinate systems. In the final section, we discuss the numerical cost of our coordinate invariant algorithm and derivations of the mathematics in sections 3 and 4 with vector operators.

BASIC EQUATIONS OF COORDINATE TRANSFORMATIONS

In this section, we recall some basic formulas of coordinate transformations. We consider two coordinate systems: an "old" system and a "new" system. The position in an old coordinate system is specified by

$$\mathbf{x} = [x_1,x_2,x_3]^T \quad . \tag{1}$$

The symbol T indicates a transpose. In our definitions of elastic and electromagnetic wave equations, the subscript notation for vectors and tensors as well as the Einstein summation convention (also known as a summation over repeated indices) will be used. Lowercase Latin subscripts are employed for this purpose (e.g., v_k, τ_{pq}); they are to be assigned the values 1, 2, and 3. Boldface symbols (e.g., $\mathbf{v}$, $\boldsymbol{\tau}$) will be used to indicate vectors or tensors. The position in the new coordinate system is specified by

$$\mathbf{x}' = [x_1',x_2',x_3']^T \quad , \tag{2}$$

with $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, or more explicitly, $x_i' = x_i'(x_j)$. We will use the prime and tilde symbols to indicate fields and physical properties in the new coordinate system (e.g., v_k', τ_{pq}', $\tilde{\mu}_0^{(ab)}$ or $\mathbf{v}'$, $\boldsymbol{\tau}'$, $\tilde{\boldsymbol{\mu}}_0$). We assume that the transformation from the old system to a new system [i.e., $\mathbf{x} = \mathbf{x}(\mathbf{x}')$, or more explicitly, $x_i = x_i(x_j')$], is uniquely defined.

Let us define the Jacobian matrix for the coordinate transformation from the old coordinate system to the new one. We denote this Jacobian matrix as $\mathbf{A}$, and its elements are defined as follows:

$$A_{ij} = \partial x_i'/\partial x_j \quad . \tag{3}$$

We assume that the Jacobian matrix is nonsingular. The Jacobian matrix of the reciprocal transform is denoted $\mathbf{A}'$, and its elements are (Post, 1962)

$$A'_{ij} = \partial x_i/\partial x'_j = \partial \ln(\alpha)/\partial A_{ij} \ , \tag{4}$$

where

$$\alpha = \det(\mathbf{A}) = \epsilon_{ijk}(\partial x'_1/\partial x_i)(\partial x'_2/\partial x_j)(\partial x'_3/\partial x_k) = \epsilon_{ijk}A_{1i}A_{2j}A_{3k} \ , \tag{5}$$

and where ϵ_{ijk} is the Levi-Civita symbol ($\epsilon_{ijk} = 1$ if ijk is an even permutation, $\epsilon_{ijk} = -1$ if ijk is an odd permutation, and $\epsilon_{ijk} = 0$ otherwise). We also have the classical identities

$$\partial/\partial x_i = (\partial x'_j/\partial x_i)(\partial/\partial x'_j) \ , \tag{6}$$

$$v_i = (\partial x'_j/\partial x_i)v'_j = A_{ji}v'_j \ , \tag{7}$$

where v_i and v'_j are components of the vectors **v** and **v**$'$, respectively. Note that in some literature **A**$'$ is considered the Jacobian matrix and **A** as the inverse. If we swap the labels "old" and "new" in the coordinate systems, the same matrices will play opposite roles. Therefore we have adopted one of the two definitions and stayed with it throughout this paper.

To add more concreteness to our definitions of Jacobian matrices, let us consider the particular case of a transformation from Cartesian coordinates to cylindrical coordinates. This transformation is defined as follows:

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \ , \\ x_3 = z \end{cases} \tag{8}$$

where x_1, x_2, and x_3 represent the old coordinate system and r, θ, and z represent the new coordinate system. The Jacobian matrices **A** and **A**$'$. for this transformation are

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -(\sin\theta)/r & (\cos\theta)/r & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{A}' = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{9}$$

The determinants of these Jacobian matrices are

$$\alpha = \det(\mathbf{A}) = 1/r \text{ and } \det(\mathbf{A}') = 1/\alpha = r \ . \tag{10}$$

COORDINATE TRANSFORMATION IN ELECTROMAGNETISM

Let **E** and **B** be the electric field and magnetic field vectors, respectively. We can define them as follows:

$$E_n \Leftrightarrow \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad \text{and} \quad H_k \Leftrightarrow \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \quad . \tag{11}$$

Using these fields, we can write the Maxwell's equations as follows [e.g., de Hoop (1995)]:

$$-\epsilon_{ijk}[\partial H_k(\mathbf{x},t,\mathbf{x}_s)/\partial x_j] + \epsilon_0^{(il)}(\mathbf{x})[\partial E_l(\mathbf{x},t,\mathbf{x}_s)/\partial x_t] = -J_i(\mathbf{x},t,\mathbf{x}_s) \quad , \tag{12}$$

$$\epsilon_{nmp}[\partial E_p(\mathbf{x},t,\mathbf{x}_s)/\partial x_m] + \mu_0^{(nr)}(\mathbf{x})[\partial H_r(\mathbf{x},t,\mathbf{x}_s)/\partial x_t] = -K_n(\mathbf{x},t,\mathbf{x}_s) \quad , \tag{13}$$

where $\epsilon_0^{(il)}(\mathbf{x})$ and $\mu_0^{(nr)}(\mathbf{x})$ are the permittivity and permeability tensors, respectively. These equations are quite general because we have considered that the permittivity and permeability can be anisotropic by describing them as second-rank tensors. The quantities **J** and **K** are the volume density of the material electric current and the volume density of the material magnetic current, respectively. In a vacuum domain, **J** and **K** are zero. The position of these sources is specified by $\mathbf{x}_s$.

Let us now show that eqs. (12) and (13) are invariant under coordinate transformation. We will start by rewriting (12) in the new coordinate system using the definition in (7) for vectors; i.e.,

$$-\epsilon_{ijk}(\partial/\partial x_j)[(\partial x_b'/\partial x_k)H_b'(\mathbf{x}',t,\mathbf{x}_s)]$$

$$+ \epsilon_0^{(il)}(\mathbf{x})[(\partial x_d'/\partial x_l)\partial E_d'(\mathbf{x}',t,\mathbf{x}_s)/\partial t] = -J_i(\mathbf{x},t,\mathbf{x}_s) \quad , \tag{14}$$

After expanding the first term on the lefthand side of (14), we arrive at

$$-\epsilon_{ijk}(\partial^2 x_b'/\partial x_j \partial x_k)H_b'(\mathbf{x}',t,\mathbf{x}_s) - \epsilon_{ijk}(\partial x_b'/\partial x_k)(\partial x_c'/\partial x_j)\partial H_b'(\mathbf{x}',t,\mathbf{x}_s)/\partial x_c'$$

$$+ \epsilon_0^{(il)}(\mathbf{x})[(\partial x_d'/\partial x_l)\partial E_d'(\mathbf{x}',t,\mathbf{x}_s)/\partial t] = -J_i(\mathbf{x},t,\mathbf{x}_s) \quad , \tag{15}$$

Notice that the first term on the lefthand side of (15) is zero. By multiplying the remaining expression by $\partial x_a'/\partial x_i$, we arrive at

$$-\epsilon_{ijk}(\partial x_a'/\partial x_i)(\partial x_b'/\partial x_k)(\partial x_c'/\partial x_j)\partial H_b'(\mathbf{x}',t,\mathbf{x}_s)/\partial x_c'$$

$$+ [(\partial x_a'/\partial x_i)\epsilon_0^{(il)}(\mathbf{x})(\partial x_d'/\partial x_l)]\partial E_d'(\mathbf{x}',t,\mathbf{x}_s)/\partial t = -(\partial x_a'/\partial x_i)J_i(\mathbf{x},t,\mathbf{x}_s) \quad . \tag{16}$$

By using the definition of the determinant of the Jacobian matrix given in (5), we can verify that

$$\epsilon_{ijk}(\partial x'_a/\partial x_i)(\partial x'_b/\partial x_k)(\partial x'_c/\partial x_j) = \alpha\epsilon_{abc} \ . \tag{17}$$

By substituting (17) into (16), we arrive at the same form of eq. (12); that is,

$$\epsilon_{bpa}[\partial H'_p(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_a] + \tilde{\epsilon}_0^{(ad)}(\mathbf{x}')\partial E'_d(\mathbf{x}',t,\mathbf{x}_s)/\partial t = -J'_a(\mathbf{x}',t,\mathbf{x}_s) \ , \tag{18}$$

where

$$\tilde{\epsilon}_0^{(ad)}(\mathbf{x}') = (1/\alpha)[(\partial x'_a/\partial x_i)\epsilon_0^{(il)}(\mathbf{x})(\partial x'_d/\partial x_l)] \ ,$$

and (19)

$$J'_a(\mathbf{x}',t,\mathbf{x}_s) = (1/\alpha)(\partial x'_a/\partial x_i)J_i(\mathbf{x},t,\mathbf{x}_s) \ .$$

Thus we see that we can interpret Ampere's law in arbitrary coordinates as the usual equation in Euclidean coordinates, as long as we use the new permittivity tensor and the new source term in (19). Using the elements of the Jacobian matrix of the reciprocal transform (3) and the results of the coordinate of second-rank tensors in (19), we can write the permittivity tensor in old coordinates as a function of the permittivity tensor in new coordinates, as follows:

$$\epsilon_0^{(ij)}(\mathbf{x}) = \alpha(\partial x_i/\partial x'_p)(\partial x_j/\partial x'_q)\tilde{\epsilon}_0^{(pq)}(\mathbf{x}') \ . \tag{20}$$

By using identical derivations, one can also show that eq. (13) can be written in the transformed coordinates as follows:

$$\epsilon_{uvw}[\partial E'_w(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_v] + \tilde{\mu}_0^{(ue)}(\mathbf{x}')\partial H'_e(\mathbf{x}',t,\mathbf{x}_s)/\partial t = -K'_u(\mathbf{x}',t,\mathbf{x}_s) \ , \tag{21}$$

where

$$\tilde{\mu}_0^{(ue)}(\mathbf{x}') = (1/\alpha)[(\partial x'_u/\partial x_n)\mu_0^{(nr)}(\mathbf{x})(\partial x'_e/\partial x_r)] \ ,$$

and (22)

$$K'_u(\mathbf{x}',t,\mathbf{x}_s) = (1/\alpha)(\partial x'_u/\partial x_n)K_n(\mathbf{x},t,\mathbf{x}_s) \ .$$

The results in (18) and (21) are simply remarkable. Variants of these equations have appeared often in the literature, such as the book on the geometry of electromagnetism by Post (1962) and the book on electromagnetic theory by Stratton (1941). These equations show that we can use the same set of Maxwell's equations for the numerical simulation of electromagnetism data, for example, irrespective of the coordinate system. We simply have to redefine the permittivity and permeability in accordance with (19) and (22). To add more concreteness to this observation, let us consider Maxwell's equations for a homogeneous isotropic medium defined by ϵ_0 and μ_0 in the Cartesian coordinate system (old system).

We can use the same Maxwell's equations in a cylindrical coordinate system (new system) as long as we replace the homogeneous isotropic medium with a heterogeneous anisotropic medium defined by the following diagonal tensors:

$$\tilde{\epsilon}_0^{(ad)}(r) = \epsilon_0 \begin{bmatrix} r & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & r \end{bmatrix} \text{ and } \tilde{\mu}_0^{(ue)}(r) = \mu_0 \begin{bmatrix} r & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & r \end{bmatrix} . \tag{23}$$

x_1', x_2' and x_3' are labeled r, θ and z, respectively. We arrive at this description of the heterogeneous anisotropic medium by substituting the elements of the Jacobian matrix in (9) into (19) and (22).

COORDINATE TRANSFORMATION IN ELASTICITY

Let τ and **v** be the stress field and the particle velocity, respectively. These quantities can be defined as follows:

$$\tau_{ij} \Leftrightarrow \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \quad \text{and} \quad v_q \Leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} . \tag{24}$$

Using these quantities, the equations of elastic wave propagation can be written as [e.g., Aki and Richards (1980), de Hoop (1995), Gangi (2000), and Ikelle and Amundsen (2005)]

$$\partial\tau_{ij}(\mathbf{x},t,\mathbf{x}_s)/\partial t = c_{ijkl}(\mathbf{x})\partial v_k(\mathbf{x},t,\mathbf{x}_s)/\partial x_l - I_{ij}(\mathbf{x},t,\mathbf{x}_s) , \tag{25}$$

$$\rho_{ir}(\mathbf{x})\partial v_r(\mathbf{x},t,\mathbf{x}_s)/\partial t - \partial\tau_{ij}(\mathbf{x},t,\mathbf{x}_s)/\partial x_j = F_i(\mathbf{x},t,\mathbf{x}_s) , \tag{26}$$

where **F** is the volume-source density of external forces and **I** is the volume-source density of the external stress-source rate; "external" forces and sources here indicate the actions of external sources on the solid under consideration. **c** is the elastic stiffness tensor of the fourth rank, and ρ_{ij} is the mass-density tensor. The stiffness tensor **c** is symmetric at each point **x**; that is, it satisfies $c_{nmpq} = c_{mnpq} = c_{mnqp} = c_{nmqp}$ in addition to $c_{nmpq} = c_{pqmn}$. We also assume that the tensorial specific volume and the stress source are symmetric; that is, $\rho_{ij} = \rho_{ji}$ and $I_{ij} = I_{ji}$ at each point **x**. Notice that we have considered in eq. (26) that the mass density can be anisotropic. Studies of composite materials have confirmed that the mass density can indeed be anisotropic.

Eq. (25) describes Hooke's law. It is known as the generalized Hooke's law because of the presence of source term I_{ij}. Eq. (26) is the Newton's equation of motion. Our objective in this section is to discuss the invariance of these two equations under coordinate transformation.

Generalized Hooke's law

Arbitrary coordinate system

Let us now discuss the invariance of eqs. (25) and (26) under coordinate transformation. We will start by rewriting (25) in the new coordinate system by using the definitions in (7) and (20) for vectors and second-rank tensors, respectively; i.e.,

$$\alpha(\partial x_i/\partial x'_p)(\partial x_j/\partial x'_q)\partial\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s)/\partial t$$

$$= c_{ijkl}(\mathbf{x})(\partial x'_d/\partial x_l)(\partial/\partial x'_d)[(\partial x'_c/\partial x_k)v'_c(\mathbf{x}',t,\mathbf{x}_s)] - I_{ij}(\mathbf{x},t,\mathbf{x}_s) \quad . \tag{27}$$

After taking the derivative of the term in the square brackets, we arrive at

$$\alpha(\partial x_i/\partial x'_p)(\partial x_j/\partial x'_q)\partial\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s)/\partial t$$

$$= c_{ijkl}(\mathbf{x})(\partial x'_d/\partial x_l)(\partial x'_c/\partial x_k)\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_d$$

$$+ c_{ijkl}(\mathbf{x})(\partial x'_d/\partial x_l)(\partial^2 x'_c/\partial x'_d\partial x_k)v'_c(\mathbf{x}',t,\mathbf{x}_s) - I_{ij}(\mathbf{x},t,\mathbf{x}_s) \quad . \tag{28}$$

We now multiply (28) by

$$(1/\alpha)(\partial x'_a/\partial x_i)(\partial x'_b/\partial x_j)$$

to obtain

$$\partial\tau'_{ab}(\mathbf{x}',t,\mathbf{x}_s)/\partial t = c'_{abcd}(\mathbf{x}')\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_d$$

$$+ d'_{abe}(\mathbf{x}')v'_e(\mathbf{x}',t,\mathbf{x}_s) - I'_{ab}(\mathbf{x}',t,\mathbf{x}_s) \quad , \tag{29}$$

where

$$c'_{abcd}(\mathbf{x}') = (1/\alpha)(\partial x'_a/\partial x_i)(\partial x'_b/\partial x_j)c_{ijkl}(\mathbf{x})(\partial x'_d/\partial x_l)(\partial x'_c/\partial x_k) \quad , \tag{30}$$

$$d'_{abe}(\mathbf{x}') = (1/\alpha)(\partial x'_a/\partial x_i)(\partial x'_b/\partial x_j)c_{ijkl}(\mathbf{x})(\partial^2 x'_e/\partial x_k\partial x_l) \quad , \tag{31}$$

$$I'_{ab}(\mathbf{x}',t,\mathbf{x}_s) = (1/\alpha)(\partial x'_a/\partial x_i)(\partial x'_b/\partial x_j)I_{ij}(\mathbf{x},t,\mathbf{x}_s) \quad . \tag{32}$$

By comparing (25) and (29), we see that the form of the stress-strain relationship varies with the coordinate system because eq. (29) contains an additional term on its lefthand side which is not present in (25). Elastic coefficients associated with this term are captured in a third-order tensor whose components are $d'_{abc}(\mathbf{x}')$. When the mapping from $\mathbf{x}$ to $\mathbf{x}'$ is such that

$$\partial^2 x'_e/\partial x_k \partial x_l = 0 \quad , \tag{33}$$

for any indices e, k, and l, then $d'_{abe}(\mathbf{x}') = 0$, and therefore the additional term in (29) is zero. In other words, the form of the stress-strain equation is invariant under coordinate transformations only for the transformations in which eq. (33) holds.

We can note that the fourth-order stiffness tensor, c'_{abcd}, still satisfies the same symmetries as c_{ijkl}; i.e.,

$$c'_{abcd} = c'_{abdc} = c'_{bacd} = c'_{cdab} \quad . \tag{34}$$

This means that the maximum number of independent stiffness constants in c'_{abcd} is 21. The third-order stiffness tensor, d'_{abe}, satisfies the following symmetries:

$$d'_{abe} = d'_{bae} \quad . \tag{35}$$

Symmetries in (35) mean that 18 is the maximum number of independent stiffness constants needed to describe this tensor in an anisotropic elastic medium.

We can also write (29) as follows:

$$\partial \tau'_{ab}/\partial t = c'_{abcd}(\mathbf{x}')(\partial w'_c/\partial x'_d) - I'_{ab}(\mathbf{x}',t,\mathbf{x}_s) \quad , \tag{36}$$

where

$$\partial w'_c/\partial x'_d = \partial v'_c/\partial x'_d + s'_{cdpq} d'_{pqr} v'_r \quad , \tag{37}$$

$$s'_{cdpq} = \alpha(\partial x_c/\partial x'_i)(\partial x_d/\partial x'_j) s_{ijkl} (\partial x_p/\partial x'_k)(\partial x_q/\partial x'_l) \quad , \tag{38}$$

and where s_{ijkl} represents the elements of the compliance tensor in the old coordinate system. Note that s'_{cdpq} represents the elements of the compliance tensor in the new coordinate system. We can verify that

$$c'_{abcd} s'_{cdpq} = \tfrac{1}{2}(\delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp}) \quad . \tag{39}$$

We can see that eq. (29) can be written in the classic form of the generalized Hooke's law in eq. (25) by constructing the special form of the particle-velocity gradient in (37). However, the form of the generalized Hooke's law in the new coordinate system in (36) is different from that of the old coordinate system because a numerical code designed for computing (25) must be modified in order to use it for the computation of (36). Notice that the system in (36) is similar to the one derived by Milton et al. (2006).

An example: Cylindrical coordinate system

Let us now look at the expressions of **c**′ and **d**′ for the cylindrical coordinates. The medium in the old coordinates is isotropic; that is

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad , \tag{40}$$

where δ_{ij} is the Kronecker delta function notation

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad i,j = 1.2.3 \tag{41}$$

and where λ and μ are the Lamé parameters. By using the elements of the Jacobian matrix in (9), we can verify that **c**′ now describes an orthorhombic medium; i.e.,

$$\begin{aligned} c'_{abcd} = {} & (1/r)[\lambda\delta_{ab}\delta_{cd} + \mu(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + (\lambda+2\mu)(r^2-1)\delta_{a1}\delta_{b1}\delta_{c1}\delta_{d1} \\ & + (\lambda+2\mu)(r^{-2}-1)\delta_{a2}\delta_{b2}\delta_{c2}\delta_{d2} + \lambda(r^2-1)(\delta_{a1}\delta_{b1}\delta_{c3}\delta_{d3} + \delta_{a3}\delta_{b3}\delta_{c1}\delta_{d1}) \\ & + \lambda(r^{-2}-1)(\delta_{a1}\delta_{b1}\delta_{c2}\delta_{d2} + \delta_{a2}\delta_{b2}\delta_{c1}\delta_{d1} + \delta_{a2}\delta_{b2}\delta_{c3}\delta_{d3} + \delta_{a3}\delta_{b3}\delta_{c2}\delta_{d2}) \\ & + \mu(r^{-2}-1)(\delta_{a1}\delta_{b2}\delta_{c1}\delta_{d2} + \delta_{a1}\delta_{b2}\delta_{c2}\delta_{d1} + \delta_{a2}\delta_{b1}\delta_{c2}\delta_{d1} + \delta_{a2}\delta_{b1}\delta_{c1}\delta_{d2}) \\ & + \mu(r^2-1)(\delta_{a1}\delta_{b3}\delta_{c1}\delta_{d3} + \delta_{a1}\delta_{b3}\delta_{c3}\delta_{d1} + \delta_{a3}\delta_{b1}\delta_{c3}\delta_{d1} + \delta_{a3}\delta_{b1}\delta_{c1}\delta_{d3}) \\ & + \mu(r^2-1)(\delta_{a2}\delta_{b3}\delta_{c2}\delta_{d3} + \delta_{a2}\delta_{b3}\delta_{c3}\delta_{d2} + \delta_{a3}\delta_{b2}\delta_{c3}\delta_{d2} + \delta_{a3}\delta_{b2}\delta_{c2}\delta_{d3})] \quad . \end{aligned} \tag{42}$$

This tensorial stiffness, c'_{abcd}, can alternatively be denoted by C'_{AB}, where subscripts A and B run from 1 to 6, with ab → A, according to 11; 22; 33; 23; 31; 12 ⇔ 1; 2; 3; 4; 5; 6. In the matrix form, it can be written

$$\mathbf{C}' = \begin{bmatrix} (\lambda+2\mu)r & \lambda r^{-1} & \lambda r & 0 & 0 & 0 \\ \lambda r^{-1} & (\lambda+2\mu)r^{-3} & \lambda r^{-1} & 0 & 0 & 0 \\ \lambda r & \lambda r^{-1} & (\lambda+2\mu)r & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu r & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu r^{-1} \end{bmatrix} . \tag{43}$$

We can see that this is a tetragonal stiffness matrix (see Ikelle and Amundsen, 2005). The third-order tensor. d'_{abe}, can also be denoted by D'_{Ae}, where the subscript A runs from 1 to 6. In the matrix form, it can be written

$$\mathbf{D}' = \begin{bmatrix} \lambda & 0 & 0 \\ (\lambda+2\mu)r^{-2} & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2\mu r^{-2} & 0 \end{bmatrix} . \tag{44}$$

By substituting (43) and (44) into (29), we can write the generalized Hooke's law in cylindrical coordinates, as follows:

$$(1/r)(\partial\tau'_{rr}/\partial t) = (\lambda + 2\mu)(\partial v'_r/\partial r) + \lambda(1/r)[(1/r)(\partial v'_\theta/\partial\theta) + v'_r] + \lambda(\partial v'_z/\partial z) - (1/r)I'_{rr} , \tag{45}$$

$$r(\partial\tau'_{\theta\theta}/\partial t) = \lambda(\partial v'_r/\partial r) + (\lambda + 2\mu)(1/r)[(1/r)(\partial v'_\theta/\partial\theta) + v'_r] + \lambda(\partial v'_z/\partial z) - rI'_{\theta\theta} , \tag{46}$$

$$(1/r)(\partial\tau'_{zz}/\partial t) = \lambda(\partial v'_r/\partial r) + \lambda(1/r)[(1/r)(\partial v'_\theta/\partial\theta) + v'_r] + (\lambda + 2\mu)(\partial v'_z/\partial z) - (1/r)I'_{zz} , \tag{47}$$

$$\partial\tau'_{r\theta}/\partial t = \mu(1/r)[(\partial v'_\theta/\partial r) + (1/r)(\partial v'_r/\partial\theta) - (2/r)v'_\theta] - I'_{r\theta} , \tag{48}$$

$$\partial\tau'_{\theta z}/\partial t = \mu(1/r)[(\partial v'_\theta/\partial z) + (\partial v'_z/\partial\theta)] - I'_{\theta z} , \tag{49}$$

$$(1/r)\partial\tau'_{rz}/\partial t = \mu[(\partial v'_z/\partial r) + (\partial v'_r/\partial z)] - (1/r)I'_{r\theta} , \tag{50}$$

with $\lambda = \lambda(r,\theta,z)$, $v'_r = v'_r(r,\theta,z,t,x_s)$, $\tau'_{rr} = \tau'_{rr}(r,\theta,z,t,x_s)$, $I'_{rr} = I'_{rr}(r,\theta,z,t,x_s)$, etc. If we introduce the quantities $\tilde{\mathbf{v}}$, $\mathbf{T}$, and $\tilde{\mathbf{I}}'$ as the particle-velocity vector, the stress tensor, and the volume-source density of the external stress-source rate, respectively, in the cylindrical coordinates and define these quantities as

$$\tilde{v}_c \Leftrightarrow \begin{bmatrix} v'_r \\ (1/r)v'_\theta \\ v_z \end{bmatrix} , \quad T_{ab} \Leftrightarrow \begin{bmatrix} (1/r)\tau'_{rr} & \tau'_{r\theta} & (1/r)\tau'_{rz} \\ \tau'_{r\theta} & r\tau'_{\theta\theta} & \tau'_{\theta z} \\ (1/r)\tau'_{rz} & \tau'_{\theta z} & (1/r)\tau'_{zz} \end{bmatrix} ,$$

and

$$\tilde{I}_{ab} \Leftrightarrow \begin{bmatrix} (1/r)I'_{rr} & I'_{r\theta} & (1/r)I'_{rz} \\ I'_{r\theta} & rI'_{\theta\theta} & I'_{\theta z} \\ (1/r)I'_{rz} & I'_{\theta z} & (1/r)\tau'_{zz} \end{bmatrix} , \tag{51}$$

we can verify that our formulation in (45)-(50) is identical to those in the literature [e.g., Liu (1999), Kausel (2006), and Pissarenko et al. (2009)]. Note that we have to redefine the stresses, particle velocities, and source terms here before comparing eqs. (45)-(50) to those encountered in this literature because their formulations are based on orthonormalized cylindrical coordinates (also known as physical coordinates), which are a particular case of cylindrical coordinates (see also Table 1). By setting $r = 1$, we obtain the Jacobian matrices $\mathbf{A}$ and $\mathbf{A}'$ in (10) for orthonormalized cylindrical coordinates. The key motivations for using orthonormalized cylindrical coordinates rather the actual cylindrical coordinates are to avoid the indefiniteness of particle velocities, stresses, strains, stiffnesses, and density. For example, we can see in (43) that c'_{1111} in the cylindrical coordinates does not correspond directly to stiffness, unlike c_{1111} in the Cartesian coordinates.

Notice that the equations in (45)-(50) are much more complicated than those in the Cartesian coordinates in (25). Yet we can rearrange eqs. (45)-(50) in a form similar to those in (25) by introducing the following new tensor for the particle-velocity gradient (strain-particle-velocity relationships):

$$\frac{\partial w'_c}{\partial x'_d} \Leftrightarrow \begin{bmatrix} \dfrac{\partial v'_r}{\partial r} & \dfrac{1}{2r}\left(\dfrac{\partial v'_\theta}{\partial r} + \dfrac{1}{r}\dfrac{\partial v'_r}{\partial \theta} - \dfrac{2}{r}v'_\theta\right) & \dfrac{1}{2}\left(\dfrac{\partial v'_z}{\partial r} + \dfrac{\partial v'_r}{\partial z}\right) \\ \dfrac{1}{2r}\left(\dfrac{\partial v'_\theta}{\partial r} + \dfrac{1}{r}\dfrac{\partial v'_r}{\partial \theta} - \dfrac{2}{r}v'_\theta\right) & \dfrac{1}{r}\left(\dfrac{1}{r}\dfrac{\partial v'_\theta}{\partial \theta} + v'_r\right) & \dfrac{1}{2r}\left(\dfrac{\partial v'_\theta}{\partial z} + \dfrac{\partial v'_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v'_z}{\partial r} + \dfrac{\partial v'_r}{\partial z}\right) & \dfrac{1}{2r}\left(\dfrac{\partial v'_\theta}{\partial z} + \dfrac{\partial v'_z}{\partial \theta}\right) & \dfrac{\partial v'_z}{\partial z} \end{bmatrix} . \tag{52}$$

Using this tensor, we can rewrite eqs. (45)-(50) as follows:

$$\partial T_{ab}/\partial t = c_{abcd}(\partial w'_c/\partial x'_d) - \tilde{I}'_{ab} , \tag{53}$$

where c_{abcd} represents the elements of the stiffness tensor in the Cartesian coordinates of the isotropic media, as defined in eq. (40). In other words, through the use of the unconventional definition of the particle-velocity gradient in (52), we can rearrange the generalized Hooke's law in cylindrical coordinates for isotropic media in a form similar to the one in equation (25), with the same stiffnesses as those in the Cartesian coordinates. Let us emphasize that, despite the fact that the form in (53) is identical to the one in (25), we consider that the form of the generalized Hooke's law in the cylindrical coordinates is different from that in the Cartesian coordinates because a numerical code designed for computing (25) must be modified for the computation of (53). For instance, the numerical code of $\partial v_c/\partial x_d$ can also be used to compute $\partial v'_c/\partial x'_d$ but not to compute $\partial w'_c/\partial x'_d$. We have found it necessary here to introduce the notations in eqs. (51)-(52) because most formulations of Hooke's law in the cylindrical coordinates for numerical computation of synthetic data in literature are in this form [e.g., Liu (1999), Kausel (2006), and Pissarenko et al. (2009)].

Equation of motion

Arbitrary coordinate system

Let us now discuss the invariance of eq. (26) under coordinate transformation. Again we will start by rewriting (25) in the new coordinate system by using the definitions in (7) and (20) for vectors and second-rank tensors, respectively; i.e.,

$$\rho_{ir}(\mathbf{x})(\partial x'_c/\partial x_r)\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial t$$
$$- (\partial x'_b/\partial x_j)(\partial/\partial x'_b)[\alpha(\partial x_i/\partial x'_p)(\partial x_j/\partial x'_q)\partial\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s)] = F_i(\mathbf{x},t,\mathbf{x}_s) \quad . \qquad (54)$$

After taking the derivative of the term in the square brackets, we arrive at

$$\rho_{ir}(\mathbf{x})(\partial x'_c/\partial x_r)\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial t$$
$$- \alpha(\partial x_i/\partial x'_p)\partial\tau'_{pb}(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_b - \alpha(\partial^2 x_i/\partial x'_b\partial x'_p)\tau'_{pb}(\mathbf{x}',t,\mathbf{x}_s)$$
$$- (\partial x_i/\partial x'_p)\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s)(\partial/\partial x_j)[\alpha(\partial x_j/\partial x'_q)] = F_i(\mathbf{x},t,\mathbf{x}_s) \quad . \qquad (55)$$

The fourth term on the lefthand side is zero because

$$(\partial/\partial x_i)[\alpha(\partial x_i/\partial x'_p)] = 0 \quad . \qquad (56)$$

By multiplying the remaining term of (55) by

$$(1/\alpha)(\partial x'_a/\partial x_i) \quad , \qquad (57)$$

we arrive at

$$\rho'_{ac}(\mathbf{x}')\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial t - \partial\tau'_{ab}(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_b$$

$$- A'_{apq}(\mathbf{x}')\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s) = F'_a(\mathbf{x}',t,\mathbf{x}_s) \ , \qquad (58)$$

where

$$\rho'_{ac}(\mathbf{x}') = (1/\alpha)(\partial x'_a/\partial x_i)\rho_{ir}(\mathbf{x})(\partial x'_c/\partial x_r) \ , \qquad (59)$$

$$F'_a(\mathbf{x}',t,\mathbf{x}_s) = (1/\alpha)(\partial x'_a/\partial x_i)F_i(\mathbf{x},t,\mathbf{x}_s) \ , \qquad (60)$$

$$A'_{apq}(\mathbf{x}') = (\partial x'_a/\partial x_i)(\partial^2 x_i/\partial x'_p/\partial x'_q) \ . \qquad (61)$$

Just like the stress-strain relationships, the Newton second relation is invariant only under coordinate transformations for the particular case in which the mapping from $\mathbf{x}$ to $\mathbf{x}'$ satisfies the condition in (33); A'_{apq} is zero for this mapping. Notice also that eq. (58) can be rewritten as follows:

$$\rho'_{ac}(\mathbf{x}')\partial v'_c(\mathbf{x}',t,\mathbf{x}_s)/\partial t - \partial T'_{ab}(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_b = F'_a(\mathbf{x}',t,\mathbf{x}_s) \ , \qquad (62)$$

where

$$\partial T'_{ab}(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_b = \partial\tau'_{ab}(\mathbf{x}',t,\mathbf{x}_s)/\partial x'_b + A'_{apq}(\mathbf{x}')\tau'_{pq}(\mathbf{x}',t,\mathbf{x}_s) \ . \qquad (63)$$

Again, these equations are quite similar to those of Milton et al. (2006).

An example: Cylindrical coordinate system

For mapping from Cartesian coordinates to cylindrical coordinates, the nonzero elements of A'_{apq} are

$$A'_{122} = -r \ , \ A'_{212} = A'_{221} = 1/r \ . \qquad (64)$$

Notice that the tensor $\boldsymbol{A}'$ is symmetric with respect to its last indices, which are p and q in eq. (61). Notice also that the density in cylindrical coordinates has the same form as the permittivity and the permeability for the same transformation in electromagnetics. By substituting (64) in (58) and assuming that the density is isotropic [i.e., $\rho_{ij}(\mathbf{x}) = \rho_0(\mathbf{x})\delta_{ij}$], we can construct the equations of motion in cylindrical coordinates, as follows:

$$r\rho_0(\partial v'_r/\partial t) - (\partial\tau'_{rr}/\partial r) - (\partial\tau'_{r\theta}/\partial\theta) - (\partial\tau'_{rz}/\partial z) + r\tau'_{\theta\theta} = F'_r \ , \qquad (65)$$

$$(1/r)\rho_0(\partial v'_\theta/\partial t) - (\partial\tau'_{\theta r}/\partial r) - (\partial\tau'_{\theta\theta}/\partial\theta) - (\partial\tau'_{\theta z}/\partial z) - (2/r)\tau'_{\theta r} = F'_\theta \ , \qquad (66)$$

$$r\rho_0(\partial v'_z/\partial t) - (\partial\tau'_{zr}/\partial r) - (\partial\tau'_{z\theta}/\partial\theta) - (\partial\tau'_{zz}/\partial z) = F'_z \ . \qquad (67)$$

By using the stress tensor **T** introduced in (51), we can rewrite these equations in the following form:

$$\rho_0(\partial v_r'/\partial t) - (\partial T_{rr}/\partial r) - (1/r)(\partial T_{r\theta}/\partial \theta) - (\partial T_{zz}/\partial z) - (1/r)(T_{rr} - T_{\theta\theta}) = (1/r)F_r' \ , \quad (68)$$

$$\rho_0(1/r)(\partial v_\theta'/\partial t) - (\partial T_{\theta r}/\partial r) - (1/r)(\partial T_{\theta\theta}/\partial \theta) - (\partial T_{\theta z}/\partial z) - (2/r)T_{\theta r} = F_\theta' \ , \quad (69)$$

$$\rho_0(\partial v_z'/\partial t) - (\partial T_{zr}/\partial r) - (1/r)(\partial T_{z\theta}/\partial \theta) - (\partial T_{zz}/\partial z) - (1/r)T_{zr} = (1/r)F_z' \ . \quad (70)$$

These are typical forms of equations of motion in cylindrical coordinates encountered in the literature [e.g., Liu (1999), Kausel (2006), and Pissarenko et al. (2009)].

Table 1. Definitions of stresses (τ_{ab}') and particle velocities (v_a') in nonorthonormal cylindrical coordinates [i.e., eq. (9)] and in orthonormal cylindrical coordinates [i.e., eq. (9), with r = 1]. The nonorthonormal cylindrical coordinates are also known as natural coordinates, and the orthonormal cylindrical coordinates are known as physical coordinates.

Orthonormal system (Physical coordinates)	Nonorthonormal systems (Natural coordinates)
$v_r' = v_1\cos\theta + v_2\sin\theta$	$v_r' = v_1\cos\theta + v_2\sin\theta$
$v_\theta' = -v_1\sin\theta + v_2\cos\theta$	$v_\theta' = (1/r)[-v_1\sin\theta + v_2\cos\theta]$
$v_z' = v_3$	$v_z' = v_3$
$\tau_{rr}' = \tau_{11}\cos^2\theta + \tau_{22}\sin^2\theta - \tau_{12}\sin(2\theta)$	$\tau_{rr}' = r[\tau_{11}\cos^2\theta + \tau_{22}\sin^2\theta - \tau_{12}\sin(2\theta)]$
$\tau_{\theta\theta}' = \tau_{11}\sin^2\theta + \tau_{22}\cos^2\theta - \tau_{12}\sin(2\theta)$	$\tau_{\theta\theta}' = (1/r)[\tau_{11}\sin^2\theta + \tau_{22}\cos^2\theta - \tau_{12}\sin(2\theta)]$
$\tau_{zz}' = \tau_{33}$	$\tau_{zz}' = r\tau_{33}$
$\tau_{\theta z}' = \tau_{23}\cos\theta - \tau_{13}\sin\theta$	$\tau_{\theta z}' = \tau_{23}\cos\theta - \tau_{13}\sin\theta$
$\tau_{rz}' = \tau_{23}\cos\theta - \tau_{13}\sin\theta$	$\tau_{rz}' = r[\tau_{23}\sin\theta + \tau_{13}\cos\theta]$
$\tau_{r\theta}' = ½(\tau_{22} - \tau_{11})\sin(2\theta) + \tau_{12}\cos(2\theta)$	$\tau_{r\theta}' = ½(\tau_{22} - \tau_{11})\sin(2\theta) + \tau_{12}\cos(2\theta)$

COORDINATE INDEPENDENT NUMERICAL MODELING

One of the main results discussed in the previous section is that (29) and (58) describe the most general form of linear elastodynamic equations. Eqs. (29) and (58) are independent of the coordinate system (see the Appendix). Hence a numerical modeling code, such as finite-difference modeling based on eqs. (29) and (58), can be used to model synthetic data in any coordinate system. The key difference between a numerical modeling code based on eqs. (29) and (58) and the present coordinate-dependent codes is that the inputs to the modeling code include the Jacobian matrix in addition to the stiffnesses and specific volumes (or densities). The flowchart in Fig. 1 illustrates how one can model eqs. (29) and (58) by using finite-difference techniques. We assume that the Cartesian coordinate system is the old coordinate system. One can alternatively use eqs. (36) and (62), which are even closer to the current FDM implementations.

Let us look at the specific case of the finite-difference modeling of eqs. (29) and (58). Because the additional terms in eqs. (29) and (58), when compared to eqs. (25) and (26), namely $d_{abe}v'_e$ and $A'_{apq}\tau'_{pq}$, do not involve differentiations of the particle velocity or stresses, the standard discretization in

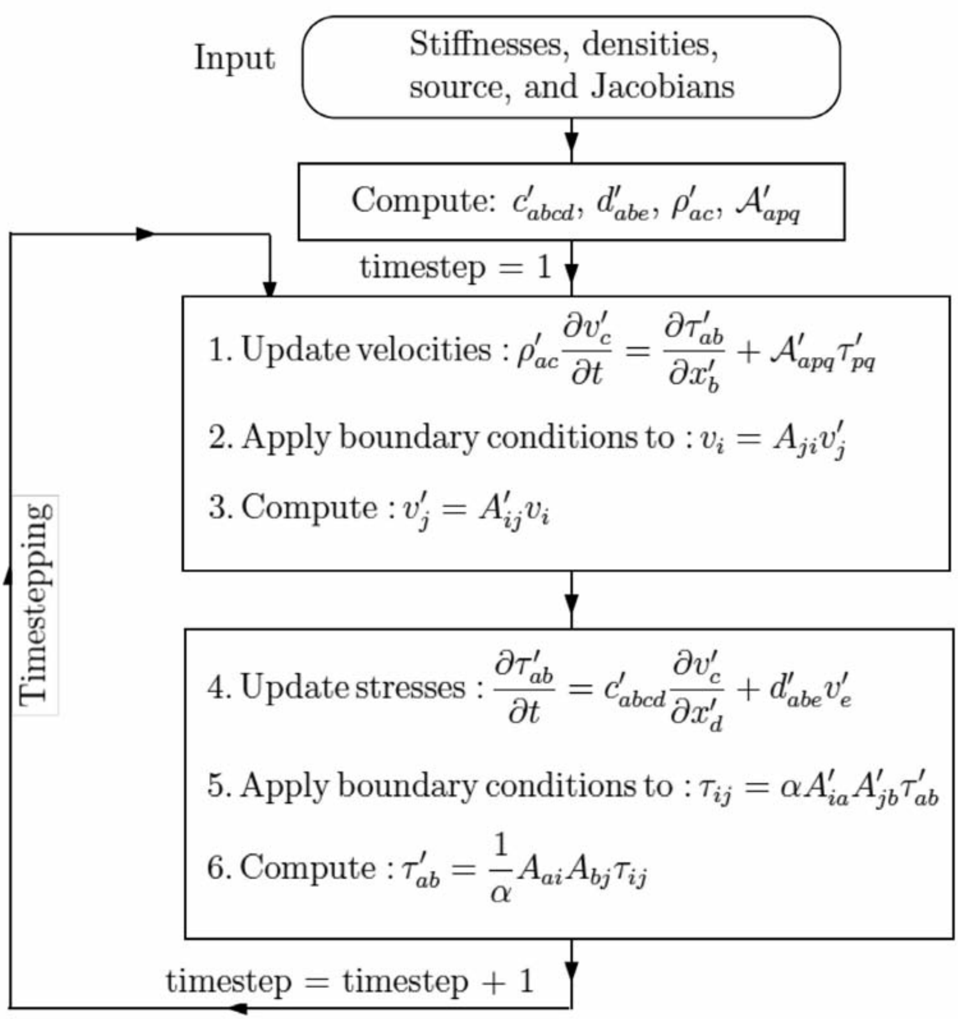

Fig. 1. A flowchart of the main steps of a numerical modeling code that is independent of coordinate transformation.

both time and space can be applied to eqs. (29) and (58) without any modifications. [See Ikelle and Amundsen (2005), Appendix C, and Ikelle (2010), Appendix C, for examples of the discretization of elastodynamic equations.] Notice that the implicit first step of this algorithm is to select a reference coordinate system. The Jacobian terms involved in these computations of (29) and (58) will be carried out with respect to the reference coordinate system. We found it useful to select a reference coordinate system for which analytic or/and numerical solutions to absorbing boundaries are known. As we will see later, this choice can facilitate the application of absorbing condition methods to any other arbitrary coordinate system. In our numerical examples, we will use the Cartesian coordinate system as the reference frame because we can take advantage of existing solutions to absorbing boundaries.

Fulfilling the stability and dispersion conditions is a major requirement for accurate finite-difference modeling. The stability condition is designed to minimize numerical errors that the timestepping (i.e., the timestep-by-timestep recursive computation) may create. The dispersion conditions are designed to minimize errors associated with the approximations of derivatives with respect to spatial coordinates. These conditions do not require any finite-difference coding; they simply requires a careful selecting of time and space sampling intervals as input to the finite-difference modeling. Moreover, the stability and dispersion conditions are well documented in the literature for various coordinate systems. They can apply to the modeling of eqs. (29) and (58) without any modifications.

Another major requirement of finite-difference modeling is the introduction of absorbing boundaries in the finite-difference code to accommodate for the fact that the subsurface is a half-space with infinite lateral boundaries. The absorbing boundary conditions can also be used for generating data without free-surface multiples by replacing the free surface with an absorbing boundary. Again, the classical solutions, such as the damping boundary conditions proposed by Cerjian (1985) [see Ikelle and Amundsen (2005), Appendix C, for the implementation], can be used to implement the absorbing boundaries, especially for Cartesian system as the reference system. The perfectly matched layer (PML) absorbing conditions described in Berenger (1994), Chew and Weedon (1994), and Chew and Liu (1996) can be used to implement the absorbing boundaries for arbitrary reference coordinate systems. The basic idea is to convert the stresses and particle velocities to reference coordinates using (7) and (19) and to apply the solution of the absorbing boundary conditions before converting the stresses and particle velocities to the actual domain. This method is valid for any anisotropic inhomogeneity medium as long as the absorbing-boundary solution for such a medium is available in the reference coordinate system. Because the damping boundary conditions proposed by Cerjian are essentially designed for Cartesian coordinate systems, one has to choose the Cartesian in this case. As shown in Liu (1999), the PML absorbing conditions can be used for Cartesian, cylindrical, and spherical coordinates. They can actually be used in many other coordinate systems. Therefore the use of PML absorbing conditions provides us several choices of the reference coordinate system.

In order to demonstrate the feasibility of the algorithm in Fig. 1, we consider a solid medium that contains a cylindrical water cavity. The source is a radial direction force with a central frequency at 100 Hz at a distance of 500 m from the origin of the grid (Fig. 2). The velocity of the P-wave in the solid is 2500 m/s, the S-wave velocity is 1300 m/s, the density is 2100 g/cm^3, the P-wave velocity in the liquid is 1500 m/s, and the density is 1000 g/cm^3. We propagated waves through this model using the algorithm in Fig. 1. We run this algorithm with and without absorbing boundary conditions. Running this algorithm without absorbing boundary conditions corresponds to ignoring steps 2, 3, 5, and 6, as defined in Fig. 1. Figs. 3a and 3b show the radial and angular particle velocity snapshots, respectively, at the same timestep for running the algorithm in Fig. 1 without absorbing boundary conditions. Figs. 4a and 4b show the same snapshots for running the algorithm in Fig. 1 with absorbing boundary conditions. We can see that the absorbing boundary condition described above and in Fig. 1 is quite effective. Notice also that the P- and S-waves are visible in these snapshots. The two wavefronts that pass through the water region are P-P, followed by S-P converted waves.

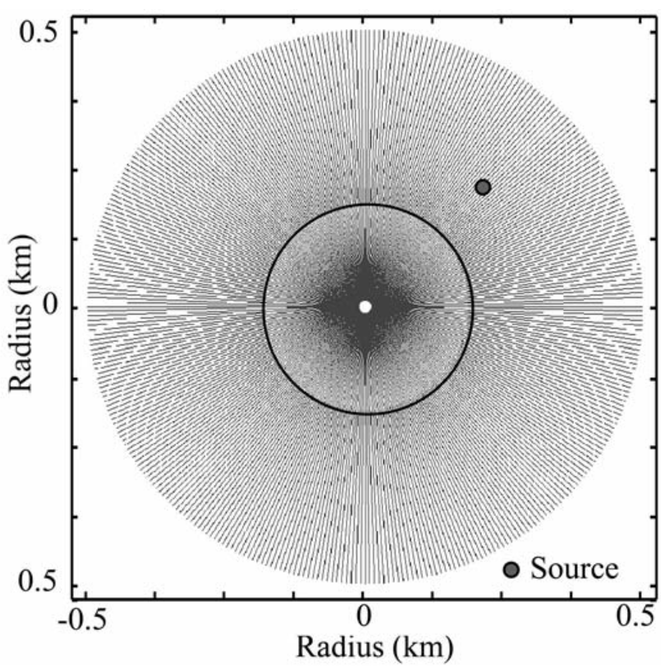

Fig. 2. Configuration of the problem. The numerical grid is composed of three parts: a water cavity, a solid medium, and an absorbing zone.

(a)

(b)

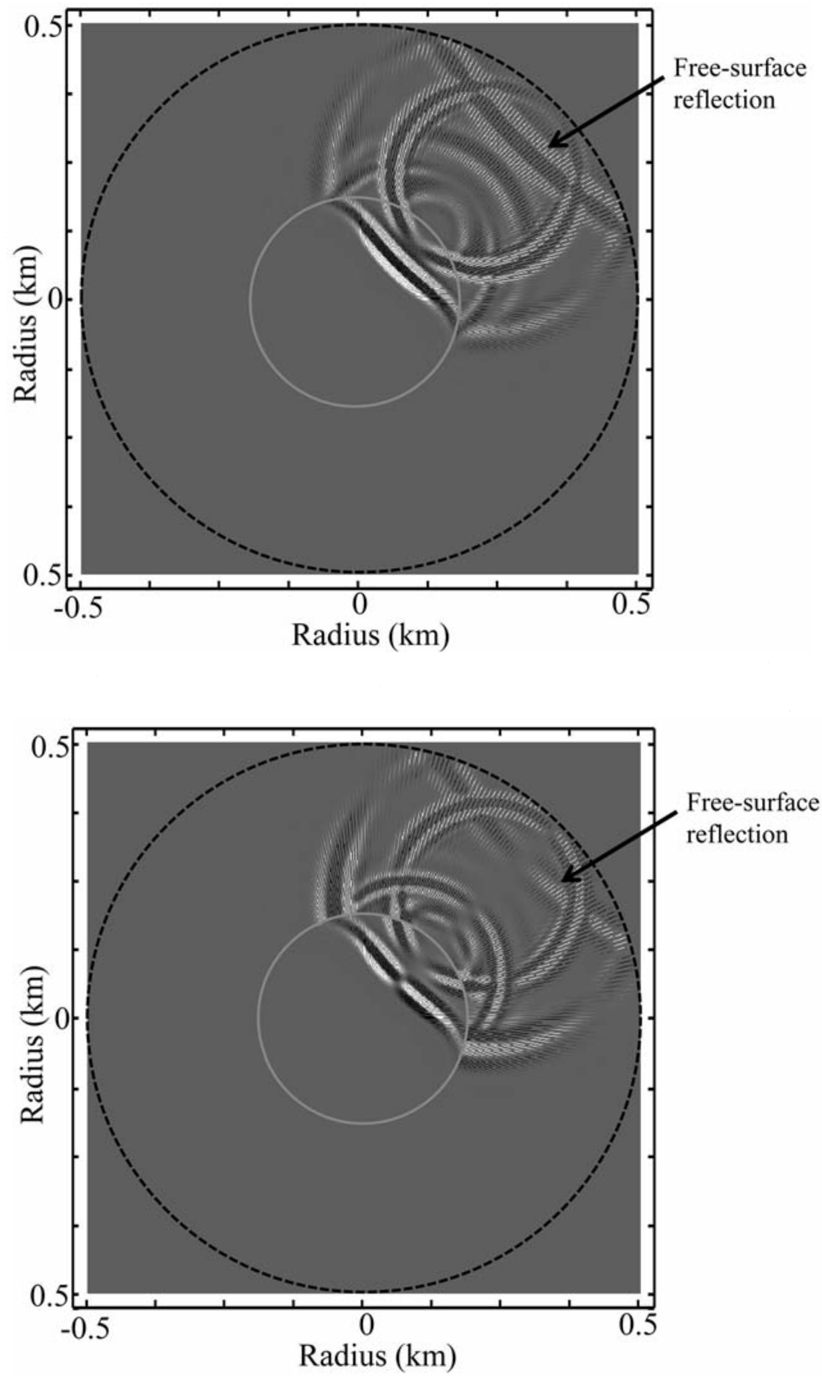

Fig. 3. Snapshots of wave propagation at time 260 ms: (a) Radial component of the particle velocity and (b) Angular component of the particle velocity. We use the algorithm in Fig. 1 without absorbing boundary conditions (i.e., without steps 2, 3, 5, and 6).

DISCUSSION

Computational cost of the coordinate independent numerical modeling

The differences in the computational cost difference between our coordinate-invariant modeling (Fig. 1) and the classical modeling in the Cartesian system (i.e., A'_{apq} and d'_{abc} are zero) are essentially related to (i) the computation of c'_{abcd}, d'_{abc}, ρ'_{ac}, and A'_{apq} and (ii) the transformation of particle velocities and stresses between Cartesian and curvilinear systems for the purpose of applying boundary condition in the Cartesian system. Because computations of c'_{abcd}, d'_{abc}, ρ'_{ac}, and A'_{apq}, are carried out outside the timestepping process, the computation time of these parameters is negligible compared to the overall computation time of the coordinate-invariant modeling, especially when a large number of timesteps, say, 1000 or more, are computed.

Let us turn to the issue of the transformation of the particle velocities and the stresses between Cartesian and curvilinear systems for the purpose of applying boundary conditions in the Cartesian system. This issue has an effect on the computation time of the coordinate-invariant modeling because these back-and-forth transformations are included in the timestepping process. We are currently developing alternative boundary conditions which do require going back to the Cartesian coordinates for their application. Hence we expect the computation time that is due to the back-and-forth transformations of the particle velocities and the stresses to disappear in our next iteration of the algorithm in Fig. 1.

In Fig. 1, we have implicitly made the assumption that our computations are taking place in a computation system in which the storage of c'_{abcd}, d'_{abc}, ρ'_{ac}, and A'_{apq} is not an issue. For computers with small storage capacities, the computation of these parameters may be included in the timestepping process to take advantage of the classical trick for reducing the storage requirements of c_{ijkl} and ρ_{ir} in standard finite-difference modeling. In most numerical modeling, the geological model is described as a set of homogeneous bodies. Each body is assigned an integer so that we can describe the geometry of the geology model by an integer array, say, $N(\mathbf{x})$. So if we have n bodies, $N(\mathbf{x})$ can only have values between 1 and n. We then define the stiffnesses and densities as follows:

$$c_{ijkl}(\mathbf{x}) = \hat{c}_{ijkl}[N(\mathbf{x})] \quad \text{and} \quad \rho_{ir}(\mathbf{x}) = \hat{\rho}_{ir}[N(\mathbf{x})] \quad , \tag{71}$$

with n being the dimension of $\hat{c}_{ijkl}$ and $\hat{\rho}_{ir}$. The inputs to the numerical modeling are $\hat{c}_{ijkl}$, $\hat{\rho}$, and $N(\mathbf{x})$ instead of $c_{ijkl}(\mathbf{x})$ and $\rho_{ir}(\mathbf{x})$. This construct of the geological models has the advantage of significantly reducing the storage requirements of c_{ijkl} and ρ_{ir}. We can still take of this construct in our coordinate independent numerical modeling by recomputing c'_{abcd}, d'_{abc}, ρ'_{ac}, and A'_{apq} at each timestep to reduce the memory requirements at the expense of increasing computation time.

(a)

(b)

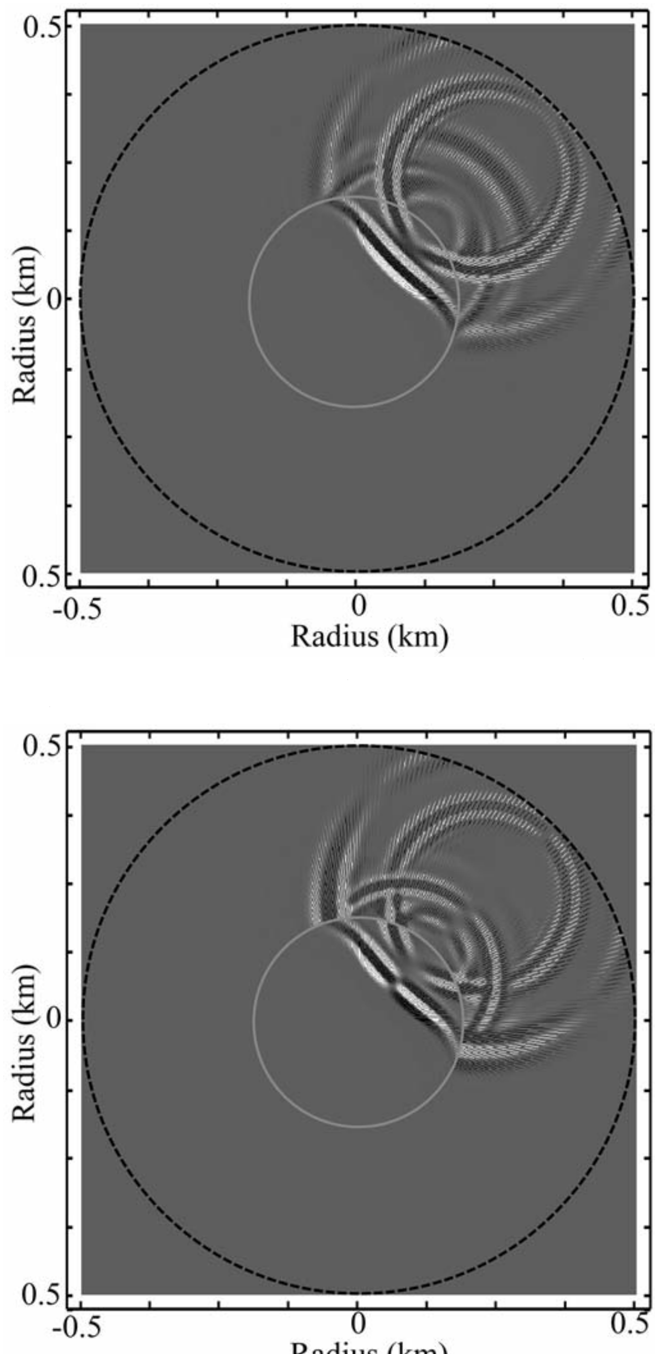

Fig. 4. Snapshots of wave propagation at time 260 ms: (a) Radial component of the particle velocity and (b) Angular component of the particle velocity. We use the algorithm in Fig. 1 with absorbing boundary conditions.

Coordinate transformation with vector notations

The Maxwell's equations in (12) and 13) can alternatively be written in vector form, as follows:

$$-\nabla \times \mathbf{H}(\mathbf{x},t,\mathbf{x}_s) + \epsilon_0(\mathbf{x})\partial\mathbf{E}(\mathbf{x},t,\mathbf{x}_s)/\partial t = -\mathbf{J}(\mathbf{x},t,\mathbf{x}_s) \ , \tag{72}$$

$$\nabla \times \mathbf{E}(\mathbf{x},t,\mathbf{x}_s) + \mu_0(\mathbf{x})\partial\mathbf{H}(\mathbf{x},t,\mathbf{x}_s)/\partial t = -\mathbf{K}(\mathbf{x},t,\mathbf{x}_s) \ . \tag{73}$$

Similarly, the equations of elastic wave propagation in (25) and (26) can alternatively be written in vector form, as follows:

$$\partial\boldsymbol{\tau}(\mathbf{x},t,\mathbf{x}_s)/\partial t - \mathbf{c}(\mathbf{x}) : \nabla\mathbf{v}(\mathbf{x},t,\mathbf{x}_s) = -\mathbf{I}(\mathbf{x},t,\mathbf{x}_s) \ , \tag{74}$$

$$\rho(\mathbf{x})\partial\mathbf{v}(\mathbf{x},t,\mathbf{x}_s)/\partial t - \nabla\cdot\boldsymbol{\tau}(\mathbf{x},t,\mathbf{x}_s) = \mathbf{F}(\mathbf{x},t,\mathbf{x}_s) \ . \tag{75}$$

We can see that the transformation of these equations from Cartesian coordinates to curvilinear coordinates is essentially based on a limited number of operations, namely the transformation operation of vectors, tensors, and derivatives of vectors, and derivatives of tensors. So one can alternatively arrive at the results presented in this paper by using a table of these operations like the one in Table 2.

Table 2. Scalar, vector, and tensor representations from the old coordinate system to the new coordinate system using the Jacobian matrix and its determinant.

Old coordinates	New coordinates
Scalar field: $\psi(\mathbf{x},t,\mathbf{x}_s)$	$\psi'(\mathbf{x}',t,\mathbf{x}_s) = \psi(\mathbf{x},t,\mathbf{x}_s)/\alpha$
Vector field: $\mathbf{E}(\mathbf{x},t,\mathbf{x}_s)$	$\mathbf{E}'(\mathbf{x}',t,\mathbf{x}_s) = [(\mathbf{A}^T)^{-1}\mathbf{E}(\mathbf{x},t,\mathbf{x}_s)]/\alpha$
Tensor field: $\boldsymbol{\tau}(\mathbf{x},t)$	$\boldsymbol{\tau}'(\mathbf{x}',t,\mathbf{x}_s) = [\mathbf{A}\boldsymbol{\tau}(\mathbf{x},t,\mathbf{x}_s)\mathbf{A}^T]/\alpha$

CONCLUSIONS

We have recast the elastodynamic equations in a new form for which the mathematical structures are invariant with the coordinate system. The numerical implementation of this new form of elastodynamic equations can be used to derive a numerical code for simulating data which are independent of the coordinate system. We have also described an absorbing boundary-condition solution for coordinate invariant numerical modeling. This solution is valid for arbitrary anisotropic inhomogeneity media and is suitable for the numerical simulation of elastic wave propagation by staggered-grid and pseudospectral finite-difference methods.

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APPENDIX

COORDINATE TRANSFORMATION

We want to show that eqs. (29) and (58) are invariant under a coordinate transformation in which $\mathbf{x}'$ now represents the "old" system and $\mathbf{x}''$ now represents the "new" system. We will use the double prime symbols to indicate fields and physical properties in the new coordinate system. By using identical derivations, as described in the previous section, one can also show that eqs. (29) and (58) can be written in the transformed coordinates, as follows:

$$\partial\tau''_{ab}(\mathbf{x}'',t,\mathbf{x}_s)/\partial t = c''_{abcd}(\mathbf{x}'')\partial v''_c(\mathbf{x}'',t,\mathbf{x}_s)/\partial x''_d + d''_{abe}(\mathbf{x}'')v''_e(\mathbf{x}'',t,\mathbf{x}_s) - I''_{ab}(\mathbf{x}'',t,\mathbf{x}_s) \quad , \tag{A-1}$$

$$\rho''_{ac}(\mathbf{x}'')\partial v''_c(\mathbf{x}'',t,\mathbf{x}_s)/\partial t - \partial\tau''_{ab}(\mathbf{x}'',t,\mathbf{x}_s)/\partial x''_b - A''_{apq}(\mathbf{x}'')\tau''_{pq}(\mathbf{x}'',t,\mathbf{x}_s) = F''_a(\mathbf{x}'',t,\mathbf{x}_s) \quad , \tag{A-2}$$

where

$$c''_{abcd}(\mathbf{x}'') = (1/\gamma)(\partial x''_a/\partial x'_i)(\partial x''_b/\partial x'_j)c'_{ijkl}(\mathbf{x}')(\partial x''_c/\partial x'_k)(\partial x''_c/\partial x'_l) \quad , \tag{A-3}$$

$$d''_{abe}(\mathbf{x}'') = (1/\gamma)(\partial x''_a/\partial x'_i)(\partial x''_b/\partial x'_j) \times [c'_{ijkl}(\mathbf{x}')(\partial^2 x''_e/\partial x'_k\partial x'_l) + d'_{ijr}(\mathbf{x}')(\partial x''_e/\partial x'_r)] \quad , \tag{A-4}$$

$$\rho''_{ac}(\mathbf{x}'') = (1/\gamma)(\partial x''_a/\partial x'_i)\rho_{ir}(\mathbf{x}')(\partial x''_c/\partial x'_r) \quad , \tag{A-5}$$

$$A''_{apq}(\mathbf{x}'') = (\partial x''_a/\partial x'_i)[(\partial^2 x'_i/\partial x''_p\partial x''_q) + A'_{iuv}(\mathbf{x}')(\partial x'_u/\partial x''_p)(\partial x'_v/\partial x''_q)] \quad , \tag{A-6}$$

$$F''_a(\mathbf{x}'',t,\mathbf{x}_s) = (1/\gamma)(\partial x''_a/\partial x'_i)F'_i(\mathbf{x}',t,\mathbf{x}_s) \quad , \tag{A-7}$$

$$I''_{ab}(\mathbf{x}'',t,\mathbf{x}_s) = (1/\gamma)(\partial x''_a/\partial x'_i)(\partial x''_b/\partial x'_j)I'_{ij}(\mathbf{x}',t,\mathbf{x}_s) \quad , \tag{A-8}$$

and where γ is the determinant of the Jacobian matrix of the coordinate transformation from $\mathbf{x}'$ to $\mathbf{x}''$. We can see that the form of eqs. (29) and (58) is preserved by this coordinate transformation.